Multi-stage Virtual Nuclear Decays

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1. INTRODUCTION

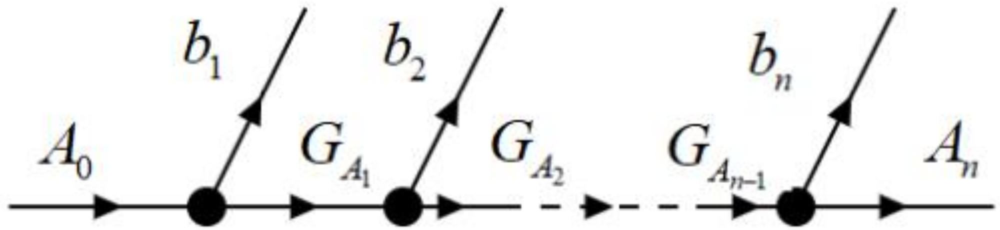
Decays and reactions with the participation of virtual intermediate states of various elementary particles are well known in the field theory [A. I. Akhiyeser and V. B. Berestetsky, *Quantum electrodynamics* (Fizmatgiz, Moscow, 1959)]. One of these reactions is the Compton scattering of γ - quanta by free electrons defined by the Feynman diagram which contains Green function describing the virtual state of intermediate electron, energy and moment of which are not connected by Einstein relativistic formula. Similar processes with the participation of analogous virtual states of different elementary particles are also known in atomic and nuclear physics.

The question arises whether there are nuclear decays and reactions with the appearance of virtual intermediate states of different atomic nuclei whose energies lie outside mass surfaces of these decays and reactions. The positive answer to this question was first given in articles [S. G. Kadmensky and Yu. V. Ivankov, Phys. Atom. Nucl. **77**, 1019 (2014); **77**, 1532 (2014)], where the theory of sequential virtual nuclear decays was developed for the description of characteristics predicted by [V. I. Goldansky, JETP **39**, 497 (1960); V. I. Goldansky, UFN **87**, 255 (1965)] and experimentally detected in [M. Pfutzner *et al*., Eur. Rev. Lett. **14**, 279 (2002); J. Giovanezzo *et al*., Phys. Rev. Let. **89**, 102501 (2002); C. Dossat *et al*., Phys. Rev. **72**, 054315 (2005); I. Mukha *et al*., Phys. Rev. Lett. **99**, 182501 (2007); I. Mukha *et al*., Phys. Rev. **77**, 061303 (2008)] $2p$ - decays of neutron-deficient nuclei. In article [D.E.Lubashevskiy, Bull. Russ. Acad. Sci., Phys, 84, 1406 (2020)] has also been demonstrated that 2β - decays of nuclei analyzed in articles [L. A. Sliv, JETP **20**, 1035 (1950); J. Suhonen, O. Civitarese, Phys. Rep. **300,** 123 (1998); V. I. Tretyak, *Double beta* *decay: history and current status*, Institute for Nuclear Research, (2014)] when using the technique of the perturbation theory have also the virtual character. In article [S. G. Kadmensky, L. V. Titova, D. E. Lubashevsky, Phys. At. Nucl., 83, 326 (2020)] the theory of virtual nuclear decays and reactions is generalized for the description of low-energy ternary fission of nuclei with the emission of long - range α - particles as third particles. Finally, in [S. G. Kadmensky, L. V. Titova, Bull. Russ. Acad. Sci., Phys., 85, 735 (2021)] the virtual mechanism was used to described the quaternary fission of nuclei with emission of two light nuclei, for example, of two α -particles The aim of this paper is to provide the comprehensive description of all known types of virtual decays of nuclei using the theory of multi-stage statistical nuclear reactions developed in the works [A. M. Lane and R. G. Thomas, Rev. of Modern Phys., 30, 257 (1958); S. G. Kadmensky, A. O. Bulychev, Bull. Russ. Acad. Sci., Phys., 79, 967 (2015); S. G. Kadmensky, A. O. Bulychev, Bull. Russ. Acad. Sci., Phys., 80, 1009 (2016)].

2. WIDTHS OF MULTI-STAGE NUCLEAR DECAYS FOR CHAINS OF GENETICALLY RELATED NUCLEI

Consider the n-step decay of the resting parent nucleus A_0 of the form $A_0 \to b_1 + A_1 \to b_1 + b_2 + A_2 \to \ldots \to b_1 + b_n + A_n$, which proceeds through a chain of decays of the states A_i of the parent (i = 0) and intermediate (i = 1,2,...n - 1) nuclei with internal E_{A_i} and kinetic T_{A_i} energies and with the appearance in the final channel of stable particles b_1, b_2,...,b_n and nucleus A_n with internal $E_{b_1}, E_{b_2}, \ldots, E_{b_n}, E_{A_n}$ and kinetic $T_{b_1}, T_{b_2}, \ldots, T_{b_n}, T_{A_n}$ energies, respectively. The amplitude of the studied decay width can be calculated on the basis of the Feynman diagram (Fig. 1), which is constructed using the diagram technique developed in [S. G. Kadmensky, A. O. Bulychev, Bull. Russ. Acad. Sci., Phys., 79, 967 (2015); S. G. Kadmensky, A. O. Bulychev, Bull. Russ. Acad. Sci., Phys., 80, 1009 (2016)].

Fig.1



In this diagram the thin lines with arrows represent the wave functions of the parent A_0, daughter A_n nuclei and particles $b_1, b_2, ..., b_n$, the vertex parts represented by the black circles connect with of the amplitudes of corresponding decay widths, and intermediate lines with arrows respond to the Green functions G_{A_i} of the intermediate nuclei defined as

$$G_{A_i} = \sum_i \frac{\left|\psi_{A_i}\right\rangle\left\langle\psi_{A_i}\right|}{Q_{A_i b_i} - T_{A_i b_i} + \frac{i\Gamma^{A_i}}{2}} \qquad (1)$$

where $T_{A_i b_i}$ - the positive definite kinetic energy of the relative motion of the particle b_i and the nucleus A_i, $Q_{A_i b_i} = \left(E_{A_{i-1}} - E_{b_i} - E_{A_i}\right)$ - the heat of decay of the nucleus A_{i-1} with the formation of the nucleus A_i and the particle b_i, and Γ^{A_i} - full width of the nucleus A_i decay. Then the analyzed partial width of the *n*-step decay of the parent nucleus A_0 it can be represented [S. G. Kadmensky, A. O. Bulychev, Bull. Russ. Acad. Sci., Phys., 79, 967 (2015); S. G. Kadmensky, A. O. Bulychev, Bull. Russ. Acad. Sci., Phys., 80, 1009 (2016)]

by the formula:

$$\Gamma^{A_0}_{A_n b_1 \dots b_n} = \frac{1}{(2\pi)^{n-1}} \int \frac{\Gamma^{A_0}_{A_1 b_1} \Gamma^{A_1}_{A_2 b_2} \dots \Gamma^{A_{n-1}}_{A_n b_n} dT_{A_1 b_1} \dots dT_{A_{n-1} b_{n-1}}}{\left[\left(T_{A_1 b_1} - Q_{A_1 b_1}\right)^2 + \frac{\left(\Gamma^{A_1}\right)^2}{4}\right] \dots \left[\left(T_{A_{n-1} b_{n-1}} - Q_{A_{n-1} b_{n-1}}\right)^2 + \frac{\left(\Gamma^{A_{n-1}}\right)^2}{4}\right]} \tag{2}$$

In formula (2) terms may appear with the heats of the decays $Q_{A_i b_i}$ having positive values $Q_{A_i b_i} > 0$, which are associated with the appearance of poles in the integrand (2) for positive values of kinetic energies $T_{A_i b_i} > 0$ and therefore correspond to real decays, which are taken into account in the traditional scheme of the description of radioactive decays chains.

At the same time in (2) terms may appear that have the heats of the decays $Q_{A_i b_i}$ with negative values and so correspond to virtual states. If all the decay heats $Q_{A_i b_i}$ appearing in formula (2) are positive, then by performing the integration in the complex plane with taking into account the Cauchy theorem, it can be to obtain the formula for the partial width $\left(\Gamma^{A_0}_{A_n b_1 \ldots b_n}\right)^{\mathrm{seq}}$ of a sequential n-step decay of the nucleus A_0 involving real decays

$$\left(\Gamma^{A_0}_{A_n b_1 \ldots b_n}\right)^{\mathrm{seq}} = \frac{\Gamma^{A_0}_{A_1 b_1} \Gamma^{A_1}_{A_2 b_2} \ldots \Gamma^{A_{n-1}}_{A_n b_n}}{\Gamma^{A_1} \ldots \Gamma^{A_{n-1}}} \quad (3)$$

coinciding with the previously obtained analogous formula [S. G. Kadmensky, A. O. Bulychev, Bull. Russ. Acad. Sci., Phys., 80, 1009 (2016)] when using the method of time-dependent kinetic equations. If all the decay heats in (2), except for the heat of the last decay, have negative values, we can obtain a formula for the partial width $\left(\Gamma^{A_0}_{A_n b_1 \ldots b_n}\right)^{\mathrm{v}}$ with taking into account virtual states of intermediate nuclei:

$$\left(\Gamma^{A_0}_{A_n b_1 \ldots b_n}\right)^{\mathrm{v}} = \frac{1}{(2\pi)^{n-1}} P\int \frac{\Gamma^{A_0}_{A_1 b_1}\Gamma^{A_1}_{A_2 b_2}\ldots\Gamma^{A_{n-1}}_{A_n b_n}\, dT_{A_1 b_1}\ldots dT_{A_{n-1} b_{n-1}}}{\left[\left(T_{A_1 b_1} - Q_{A_1 b_1}\right)^2 + \frac{\left(\Gamma^{A_1}\right)^2}{4}\right]\ldots\left[\left(T_{A_{n-1} b_{n-1}} - Q_{A_{n-1} b_{n-1}}\right)^2 + \frac{\left(\Gamma^{A_{n-1}}\right)^2}{4}\right]}, \quad (4)$$

where index P means integration in the sense of the main value. In general case the considered width of the n-step decay $\Gamma^{A_0}_{A_n b_1 \ldots b_n}$ can be represented as the sum of the widths for compositions associated with different numbers of real and virtual decays. The developed above theory will be used below to describe two-stage virtual 2p, 2β - decays of nuclei, as well as ternary fission of nuclei and three-stage virtual quaternary fission of nuclei.

3. TWO-PROTON DECAYS OF NUCLEI AS VIRTUAL PROCESSES

At present the widely used approach for the description of $2p$ - decays of nuclei is based [L. V. Grigorenko *et al*., Phys. Rev. Lett. **85**, 22 (2000); L. V. Grigorenko *et al*., Phys. Rev. **64**, 054002 (2001); L. V. Grigorenko and M. V. Zhukov, Phys. Part. **76**, 014009 (2007); L. V. Grigorenko, Phys. Part. Nucl. **40**, 674 (2009)] on the idea of the simultaneous emission from the parent nuclei (A, Z) two protons with the formation of final nuclei $(A-2, Z-2)$ when the amplitudes of $2p$ - decays are presented by the Feynman diagrams Fig. 2.

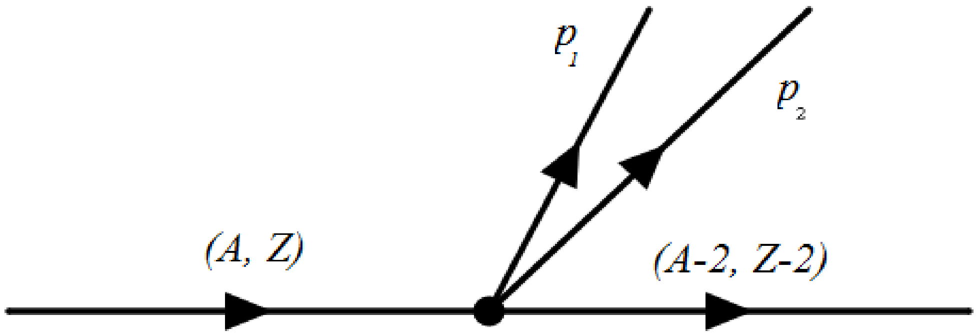
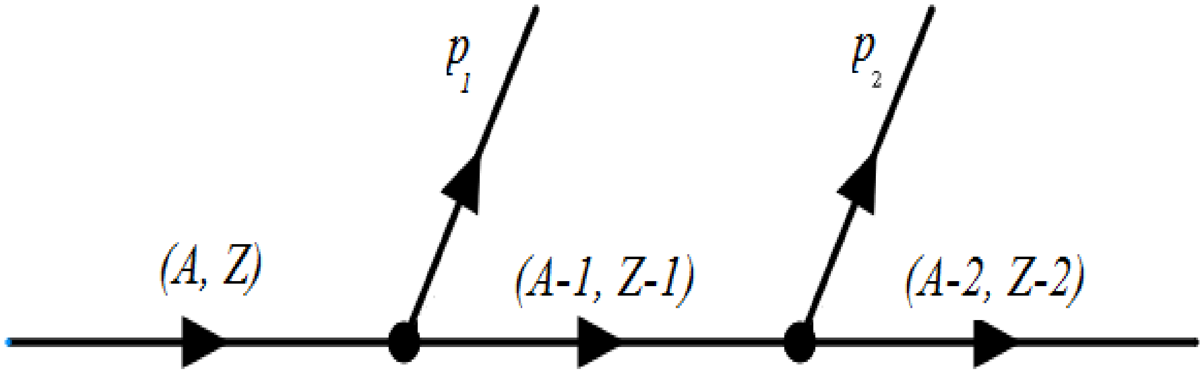


Fig. 2

The vertex part of like diagram is expressed through the amplitude of $2p$ - decay width for parent nuclei (A, Z) which is built in the framework of the theory in which the three-particle wave function of the final channel of this decay taking into account not only pair, but also three-particles potentials of interactions of emitted particles. The most consistent version of such theories was developed [L. V. Grigorenko *et al*., Phys. Rev. Lett. **85**, 22 (2000); L. V. Grigorenko *et al*., Phys. Rev. **64**, 054002 (2001); L. V. Grigorenko and M. V. Zhukov, Phys. Part. **76**, 014009 (2007); L. V. Grigorenko, Phys. Part. Nucl. **40**, 674 (2009)] on the basis of the R-matrix theory of nuclear reactions [M. Goldberger, K. Watson, *Collision Theory* (Wiley, N.Y. 1964; Mir: Moscow, 1967); A. M. Lane, R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958)] using the method of hyperspheric harmonics. The parameters of three particles potentials are determined developed [L. V. Grigorenko *et al*., Phys. Rev. Lett. **85**, 22 (2000); L. V. Grigorenko *et al*., Phys. Rev. **64**, 054002 (2001); L. V. Grigorenko and M. V. Zhukov, Phys. Part. **76**, 014009 (2007); L. V. Grigorenko, Phys. Part. Nucl. **40**, 674 (2009)] by fitting the calculated characteristics of $2p$- decays to their experimental values. However, in cited works there is no answer to the question why these potentials are not taken into account for the description of structural properties of nuclei in traditional theories [V. G. Solovyev, *The theory of the atomic nucleus: Nuclear models* (Energoatomizdat, Moscow, 1981); A. B. Migdal, *Theory of finite Fermi systems and properties of atomic nuclei*, (Mir, Moscow, 1983)], in which only pair nucleon-nuclear (shell) and nucleon-nucleon effective potentials are used.

In [S. G. Kadmensky and Yu. V. Ivankov, Phys. Atom. Nucl. **77**, 1019 (2014); S. G. Kadmensky and Yu. V. Ivankov, Phys. Atom. Nucl. **77**, 1532 (2014); S. G. Kadmensky, Yu. V. Ivankov and D. E. Lyubashevsky,Phys. Atom. Nucl. **80**, 903 (2017)] the theory of two-stage virtual nuclear decays was developed on the example of $2p$-decays, which are described as the two-stage processes by the Feynman diagram Fig. 3:

Fig. 3



where due to the negativity of the heat of the single-proton decay of the nucleus (A, Z) the internal line corresponds to the Green function of the intermediate nucleus (A-1, Z-1) being in the virtual state. The vertex parts of Fig. 3 are expressed [S. G. Kadmensky and Yu. V. Ivankov, Phys. Atom. Nucl. **77**, 1019 (2014); S. G. Kadmensky and Yu. V. Ivankov, Phys. Atom. Nucl. **77**, 1532 (2014)] through the amplitudes of single-proton decay widths for parent (A, Z) and intermediate $(A-1, Z-1)_i$ nuclei determined by integral formulae [S. G. Kadmensky and V. I. Furman, *Alpha decay and related nuclear reactions,* (Energoatomizdat, Moscow, 1985); S. G. Kadmensky*et al*, Phys. Atom. Nucl. **42**, 1123 (1985); V. P. Bugrov and S.G. Kadmensky, Sov. Jour. Nucl. Phys. **49**, 156 (1989)]. In these formulae the shell model potentials and effective potentials of nucleon-nucleon interaction are used as in generally accepted multi-particle theories of the nucleus [V. G. Solovyev, *The theory of the atomic nucleus: Nuclear models* (Energoatomizdat, Moscow, 1981); A. B. Migdal, *Theory of finite Fermi systems and properties of atomic nuclei,* (Mir, Moscow, 1983)] without taking into account three-particle interactions.

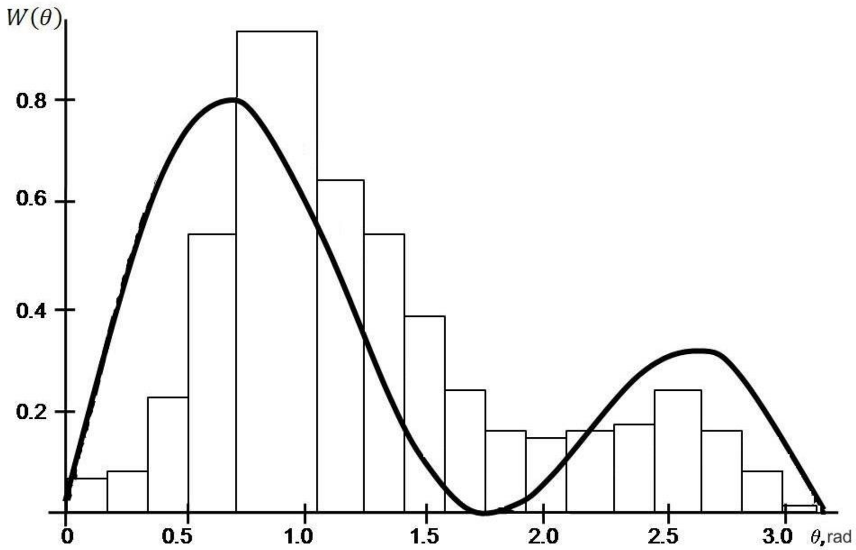
As it can be seen from Table 1 the experimental values of $2p$ – widths of parent nuclei ^{16}Ne, ^{19}Mg, ^{45}Fe, ^{48}Ni, ^{67}Kr are in the reasonable agreement with theoretical values of analogous widths, calculated within the framework of the superfluid model of atomic nuclei [V. G. Solovyev, *The theory of the atomic nucleus: Nuclear models* (Energoatomizdat, Moscow, 1981)].

Table 1. Characteristics of virtual $2p$ - decays of nuclei.

№	Decay $(A, Z) \rightarrow (A-2, Z-2)$	A	Z	Exp. width of $2p$ - decay $(\Gamma_{2p}^{A})^{\mathrm{exp}}$, MeV	Th. width of $2p$ - decay $(\Gamma_{2p}^{A})^{\mathrm{v}}$, MeV
1	Ne $\rightarrow$ O	16	10	$1{,}11^{+0{,}01}_{-0{,}01} \cdot 10^{-3}$	$1{,}2 \cdot 10^{-3}$
2	Mg $\rightarrow$ Ne	19	12	$1{,}1^{+1{,}4}_{-0{,}25} \cdot 10^{-10}$	$2{,}4 \cdot 10^{-10}$
3	Fe $\rightarrow$ Cr	45	26	$1{,}6^{+0{,}5}_{-0{,}5} \cdot 10^{-19}$	$1{,}8 \cdot 10^{-19}$
4	Ni $\rightarrow$ Fe	48	28	$2{,}2^{+1{,}1}_{-1{,}1} \cdot 10^{-19}$	$1{,}6 \cdot 10^{-19}$
5	Kr $\rightarrow$ Se	67	36	$0{,}62^{+0{,}03}_{-0{,}03} \cdot 10^{-19}$	$0{,}62 \cdot 10^{-19}$

As it is seen on Fig.4 there is also the agreement of the experimental angular distribution $W^{\mathrm{exp}}(\theta)$ [K. Miernik *et al*., Phys. Rev. Lett. **72**, 054315 (2005); K. Miernik *et al*., Nucl. Instr. Meth. Phys. Sect. A. **581**, 194 (2007)], where (θ) is the angle between directions of flights of first and second protons for $2p$ - decay of ^{45}Fe, and of the analogous theoretical distribution calculated in [S. G. Kadmensky, Yu. V. Ivankov and D. E. Lyubashesky, Phys. Atom. Nucl. **80**, 903 (2017)] with structural parameters used earlier in the calculation of $2p$- decay width of ^{45}Fe, presented in the Table 1.

Fig. 4

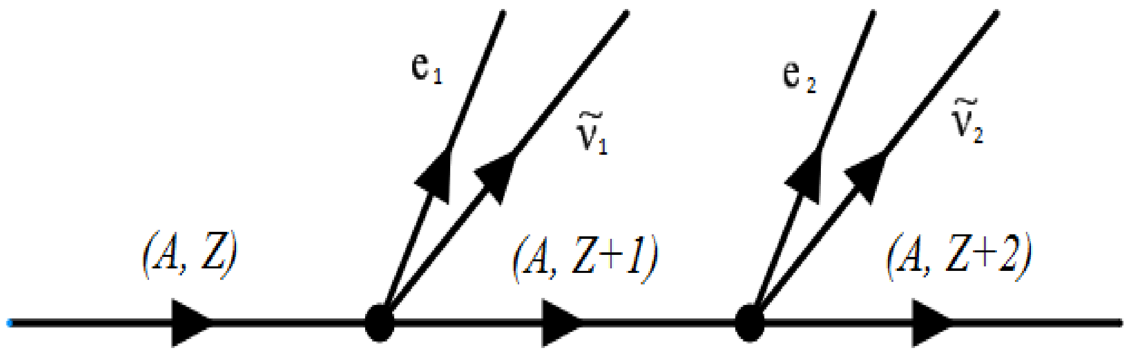


4. DOUBLE β –DECAYS OF NUCLEI AS VIRTUAL AND REAL PROCESSES

The first version of the theory of 2β–decays of nuclei was constructed [L. A. Sliv, JETP **20**, 1035 (1950)] on the example of $2\beta^{-}$- decays with the usage of the formula for probabilities $\omega_{2\beta^{-}}$ of the investigated decays per unit time determined in the second order of the perturbation theory by the Hamiltonian H' of the weak interaction. The formalism of article [L. A. Sliv, JETP **20**, 1035 (1950)] is generalized [J. Suhonen, O. Civitarese, Phys. Rep. **300,** 123 (1998); V. I. Tretyak, *Double beta decay: history and current status,* Institute for Nuclear Research, (2014)] for the usage of more modern versions of weak interactions.

From (5-7) it follows that the considered $2\beta^-$- decays can be presented by the analogy with named above $2p$- decays as the virtual two-stage processes described by the Feynman diagram Fig. 5. The vertex parts of this diagram are connected with matrix elements (7)

Fig. 5



In Table 2 experimental characteristics [J. Suhonen, O. Civitarese, Phys. Rep. **300,** 123 (1998); V. I. Tretyak, *Double beta decay: history and current status*, Institute for Nuclear Research, (2014); B. S. Ishkhanov, *Radioactivity* (University book, Moscow 2011)] of $2\beta^-$-decays of the ground states of a large group of parent even-even nuclei are presented. In all reviewed cases experimental half-lives have so large values $T_{1/2}^{\exp} > 10^{17}$ years that they exceed the lifetime of the Universe (≈10^{10} years).Since values of heats $Q_{\beta_1^- i}$ for β_1^-- decays of parent nuclei have negative values, all $2\beta^-$- decays considered in Table 2 are two stage virtual decays. As can be seen from table 2, the values of $T_{1/2}^{\text{th}}$ [J. Suhonen, O. Civitarese, Phys. Rep. **300,** 123 (1998); V. I. Tretyak, *Double beta decay: history and current status*, Institute for Nuclear Research, (2014); B. S. Ishkhanov, *Radioactivity* (University book, Moscow2011)] for most nuclei are reasonably consistent with similar values of $T_{1/2}^{\exp}$.

Table 2. Characteristics of virtual $2\beta^-$ - decays of nuclei.

		A	*Z*	$Q_{2\beta^-}$ (*A*,*Z*) → (*A*,*Z*+*2*), keV	$T_{1/2}^{\exp}(2\beta^-)$, years	$T_{1/2}^{\exp}(2\beta^-)$, years	Q_{β_1} - (*A*,*Z*)→ (*A*,*Z*+*1*), keV	$T_{1/2}^{\exp}(\beta_1^-)$, years
1	Ca→Ti	48	20	4271,7±5,4	1,9×10^{19}	2,6×10^{19}	+281±6	1,9×10^{19}
2	Ge→Se	76	32	2045,7±5	1,6×10^{21}	8,5×10^{20}	-922,9±2,7	
3	Se→Kr	82	34	3005±16	9,2×10^{19}	6,7×10^{19}	-88±12	
4	Zr→Mo	96	40	3350,2±6,1	2,0×10^{19}	1,3×10^{20}	+163,0±5	>3,8×10^{19}
5	Mo→Ru	100	42	3032,6±8,6	7,3×10^{18}	3,2×10^{19}	-170±6	
6	Pd→Cd	110	46	2014±24	>6×10^{17}	6,3×10^{20}	-879±20	
7	Cd→Sn	116	48	2808,5±7,3	3,3×10^{19}	7,3×10^{19}	-464±8	
8	Sn→Te	124	50	2278,3±8,8	>1,2×10^{21}	1,5×10^{21}	-627±5	
9	Te→Xe	128	52	868,9±5,5	2,41×10^{24}	1,6×10^{24}	-1258±5	
10	Te→Xe	130	52	2533,1±6,6	6,9×10^{20}	4×10^{20}	-451±11	
11	Xe→Ba	136	54	2481±15	2,2×10^{21}	4,5×10^{20}	-67±11	
12	Nd→Sm	148	60	1928±10	>3×10^{18}	1×10^{21}	-536±9	
13	Nd→Sm	150	60	3367±11	8,2×10^{18}	5,8×10^{18}	-130±80	
14	Sm→Gd	154	62	1250±10	>2,3×10^{18}	1,49×10^{22}	-728±5	
15	Gd→Dy	160	64	1731±11	>1,9×10^{19}	7,2×10^{20}	-102,3±1,4	
16	U→Pu	238	92	1146,2±4,6	2×10^{21}	1,9×10^{22}	-145,6±1,3	

However, for nuclei ^{110}Pd, ^{148}Nd and ^{154}Sm differences between $T_{1/2}^{\mathrm{exp}}$ and $T_{1/2}^{\mathrm{th}}$ reach three or more orders, which is apparently due to significant inaccuracies in the determination of experimental values of $T_{1/2}^{\mathrm{exp}}$ for these nuclei.

In Table 3 experimental characteristics [B. S. Ishkhanov, *Radioactivity* (University book, Moscow2011);https://www-nds.iaea.org/relnsd/vcharthtml/VChartHTML.html; https://www.nndc.bnl.gov/.;http://nucleardata.nuclear.lu.se/toi/nucSearch.asp.; http://cdfe.sinp.msu.ru/services/gsp.ru.html.; M. Hirsch et al. Z. Phys. A **347**, 151 (1994)] of $2\beta^{+}$ - decays of a group of nuclei are presented. Because of the heats $Q_{\beta_1^{+}i}$ for β_1^{+} - decays of parent nuclei have negative values, the considered $2\beta^{+}$ - decays have the virtual character. As can be seen from table 3, experimental values of $T_{1/2}^{\mathrm{exp}}$ and calculated in work [M. Hirsch et al. Z. Phys. A **347**, 151 (1994)] the values of $T_{1/2}^{\mathrm{th}}$ for all considered nuclei differ by more than 10 orders of magnitude, that may be due to very large inaccuracies in the determination of experimental values of $T_{1/2}^{\exp}$.

Table 3. Characteristics of virtual $2\beta^{+}$ - decays of nuclei.

№	Decay $(A,Z) \to (A,Z\text{-}2)$	A	Z	Heat $Q_{2\beta^+}$ of the $2\beta^+$- decay $(A,Z) \to (A,Z\text{-}2)$, keV	Exp. half-life $T_{1/2}^{\exp}$ of the $2\beta^+$-decay, years	Th. half-life $T_{1/2}^{\text{th}}$ of the $2\beta^+$-decay, years	Heat $Q_{\beta_1^+}$ of the β_1^+- transition $(A,Z) \to (A,Z\text{-}1)$, keV
1	Ru→Mo	96	44	2719,9±11,4	>3,1×10^16	5,8×10^26	-254±10
2	Pd→Ru	102	46	1175,5±11,9	-	2,5×10^32	-1148±6
3	Cd→Pd	106	48	2782±11	>2,4×10^20	4,6×10^26	-202±9
4	Xe→Te	124	54	3068,3±143,8	2,0×10^14	1,4×10^27	-90±140
5	Ba→Xe	130	56	2578,1±13,6	-	1,7×10^29	-440,9±3,9
6	Ba→Xe	132	56	833±15	3,0×10^21	-	-1279±24
7	Os→W	184	76	1454±14	5,6×10^13	-	-42±6

In Table 4 experimental half-lives of $T_{1/2}^{\mathrm{exp}}$ [https://www-nds.iaea.org/.; https://www.nndc.bnl.gov/.; http://cdfe.sinp.msu.ru/services/gsp.ru.html.; M. Hirsch et al. Z. Phys. A **347**, 151 (1994)] for (β^+, EC) and (EC, EC) decays of a group of the ground state nuclei are presented. Because of the heats Q_{β^+} for β^+- decays and Q_{EC} for electron captures in these nuclei have negative values the considered (β^+, EC) and (EC, EC) decays have the virtual character. As can be seen from table 4, almost for all (β^+, EC) and (EC, EC)- decays the values of experimental half-lives $T_{1/2}^{\mathrm{exp}}$ and analogous theoretical half-lives $T_{1/2}^{\mathrm{th}}$, calculated in work [M. Hirsch et al. Z. Phys. A **347**, 151 (1994)], differ by more than 10 orders of magnitude. Only for (EC, EC)- decay of nucleus ^{130}Ba the certain agreement between the values $T_{1/2}^{\mathrm{exp}}$ and $T_{1/2}^{\mathrm{th}}$ is observed.

Table 4. Characteristics of virtual (β^+, EC) and (EC, EC) decays.

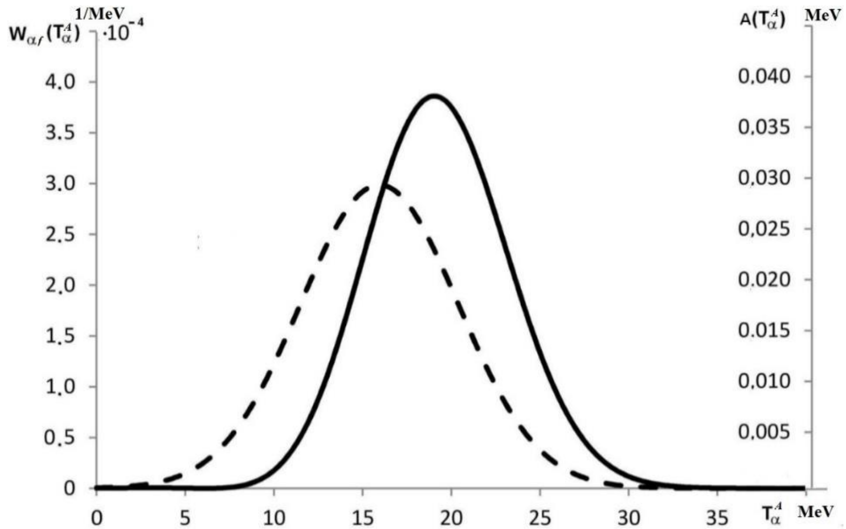
Isotope	Heat Q_{β^+} of the β^+-transition $(A,Z)\rightarrow(A,Z\text{-}1)$, keV	$T_{1/2}^{\exp}(\beta^+, EC)$, years	$T_{1/2}^{\mathrm{th}}(\beta^+, EC)$, years	Heat Q_{EC} of the EC -transition $(A,Z)\rightarrow(A,Z\text{-}1)$, keV	$T_{1/2}^{\exp}(EC, EC)$, years	$T_{1/2}^{\mathrm{th}}(EC, EC)$, years
^{58}Ni	-380	>6,2×10^{19}	5,5×10^{25}	-1402	>4×10^{19}	3,9×10^{24}
^{78}Kr	-727	-	5,3×10^{22}	-1749	>2,3×10^{20}	3,7×10^{22}
^{96}Ru	-254	>6,7×10^{16}	1,2×10^{22}	-1276	-	2,1×10^{21}
^{106}Cd	-194	>1,2×10^{18}	4,1×10^{21}	-1216	>1,0×10^{18}	8,7×10^{20}
^{124}Xe	-296	>4,8×10^{16}	3,0×10^{22}	-1318	>1,1×10^{17}	2,9×10^{21}
^{130}Ba	-361	-	1,0×10^{23}	-1383	2,2×10^{21}	4,2×10^{21}

For the two parent nuclei $^{48}Ca_{20}$ and $^{96}Zr_{40}$, presented in the Table. 2, the energies Q_{β_1} turn out to be positive, what makes open not only the channels of $2\beta^-$ - decay of the parent nucleus A_0 and the decay of the intermediate nucleus A_1, but also $\beta^-$the channel of decay of the parent nucleus A_0. This means that this variant of sequential two-stage-decay could not be taken into account in earlier theoretical works [L. A. Sliv, JETP **20**, 1035 (1950); J. Suhonen, O. Civitarese, Phys. Rep. **300,** 123 (1998); V. I. Tretyak, *Double beta decay: history and current status*, Institute for Nuclear Research, (2014)] based on the second order of the perturbation theory according to the weak interaction Hamiltonian. Then we have two real β^- decays following each other in time, and when using formula (3) taking into account, that the total width Γ^{A_1} of the decay of the nucleus A_1 coincides with the $\Gamma^{A_1}_{A_2\beta_2^-}$ - partial width of its decay, we conclude that the partial width $\Gamma^{A_0}_{A_2 2\beta^-}$ for the $2\beta^-$ decay of the nucleus A_0 coincides with the partial width $\Gamma^{A_0}_{A_1\beta_1^-}$. This result is fairly well reflected in Table 2.

5. TERNARY FISSION OF ATOMIC NUCLEI AS BINARY VIRTUAL PROCESS

Using the existing information [A. Bohr and B. Mottelson, Nuclear Structure (Benjamin, New York, 1969, 1975), Vol. 1, 2; M. Mutterer and J. P. Theobald, Dinuclear Decay Modes, Bristol: IOP Publ., 1996, Chap. 12; O. Tanimura, T. Fliessbach, Z. Phys. **328**, 475 (1987)] about properties of binary and ternary nuclear fission, let us analyze mechanisms of the ternary spontaneous fission of parent nucleus (A, Z) with the emission of two primary light (A_{LF}, Z_{LF}) and heavy (A_{HF}, Z_{HF}) fission fragments and third light particle as which we consider α- particle. The characteristics of the ternary fission α - particles significantly differ from the characteristics of α - particles flying in the standard favourable α - decays of parent nuclei from the first well of their deformation potential. The heats of these decays $Q_\alpha^A \approx (5-6)$ MeV noticeably less the most probable energies $(T_\alpha^A)_0 \approx 16$MeV in the experimental energy distribution $W_{\alpha f}^{\exp}(T_\alpha^A)$ of α-particles for spontaneous ternary fission of nuclei [S. Vermote *et al*.,Nucl. Phys. A. **837**, 176 (2010)]. The distribution $W_{\alpha f}^{\exp}(T_\alpha^A)$ is presented for ^{252}Cf on fig. 7.

Fig. 7



In contrast to the isotropic angular distribution of α - particles emitted during the α - decay of non-spin-oriented parent nuclei, the experimental angular distribution $W_{\alpha f}^{\exp}(\theta)$ of α- particles of the spontaneous ternary fission where θ is the angle between directions of emissions of the α - particle and the light fission fragment, is anisotropic [S. Vermote *et al*.,Nucl. Phys. A. **837**, 176 (2010)]. The maximum of this distribution is formed at the angle $\theta \approx \pi/2$.

Nowadays there are several mechanisms of the ternary nuclear fission discussed in the review [M. Mutterer and J. P. Theobald, Dinuclear Decay Modes, Bristol: IOP Publ., 1996, Chap. 12]. The evaporative mechanism presented in some early works and associated with the emission of α-particles from heated to high temperatures fissile nuclei is not realized since fissile nuclei at all stages of their evolution remain in cold non-thermalized states, which in the vicinity of the scission point transform into transitional fission states [A. Bohr and B. Mottelson, Nuclear Structure (Benjamin, New York, 1969, 1975), Vol. 1, 2]. The most interesting among discussed mechanisms is the nonstationary nonadiabatic mechanism [O. Tanimura, T. Fliessbach, Z. Phys. **328**, 475 (1987)] which allows to take into account the influence on emitted α - particle of the time-dependent potential of its interaction with residual nucleus $(A-4, Z-2)$. Here with it is assumed too that the α-particle is formed not in the first well of the deformation potential of fissile nucleus, but in its configurations denoted further by the symbol (0). These configurations are achieved by fissile nucleus with the probability ω_0^{A} after its transition through the first and second fission barriers and the appearance of the pear shape of this nucleus corresponding to two deformed fission pref-ragments connected by a neck. Under the action of the named above nonadiabatic potential the emitted α - particle increases its asymptotical kinetic energy T_α^A in the comparison with the analogous energy Q_α^A of the α - decay α - particle. The influence of Coulomb interaction of the ejectedα-particle with light and height fission fragments having different electric charges, leads to the appearance of named above anisotropy for the angular distribution of this particle. Thus, the nonadiabatic mechanism of the ternary fission allows to explain the named above differences in the characteristics of α- particles for ternary fission and α - decay of parent nuclei.

Unfortunately, the nonadiabatic mechanism because of its nonstationarity does not take into account the energy conservation law (16) for the closed system of the fissile nucleus. In this case the increase ofα - particle kinetic energy T_{α}^{A} in comparison to the heat Q_{α}^{A}must lead to the adequate decrease of the asymptotic kinetic energy of the relative motion T_{f}^{A-4} for fragments of the residual nucleus$(A-4, Z-2)$ fission. For this reason it can propose the new stationary approach for the description of spontaneous ternary fission of nuclei based on the two-stage virtual mechanism of this fission similar to the analogous mechanism of $2p$-decay of nuclei [S. G. Kadmensky and Yu. V. Ivankov, Phys. Atom. Nucl. **77**, 1019 (2014); S. G. Kadmensky and Yu. V. Ivankov, Phys. Atom. Nucl. **77**, 1532 (2014)]. The formula for the width $(\Gamma_{\alpha f}^{A})^{\mathrm{v}}$ of the virtual spontaneous ternary fission can be presented by the analogy with formula (4) as

$$(\Gamma_{\alpha f}^{A})^{\mathrm{v}} = \frac{1}{2\pi} \int_{Q_{\alpha}^{A}+\Delta}^{Q_{\alpha f}^{A}} \frac{\bar{\Gamma}_{\alpha}^{A}(T_{\alpha}^{A})\Gamma_{f}^{A-4}\left(T_{f}^{A-4}\right)}{(Q_{\alpha}^{A}-T_{\alpha}^{A})^{2}}\, dT_{\alpha}^{A}\,, \qquad (5)$$

where $\bar{\Gamma}_{\alpha}^{A}(T_{\alpha}^{A})$ is the width of the virtual α - decay of the parent nucleus (A, Z) connected with its transition of this nucleus with the probability ω_{0}^{A} to its configuration (0) and the subsequent α - decay of named above nucleus from this configuration with the width $\Gamma_{\alpha}^{A}(T_{\alpha}^{A})$

$$\bar{\Gamma}_{\alpha}^{A}(T_{\alpha}^{A}) = \omega_{0}^{\mathrm{A}} \cdot \Gamma_{\alpha}^{A}(T_{\alpha}^{A})\,. \qquad (6)$$

Using the experimental characteristics of ternary spontaneous fission of nuclei presented in Table 5 and the experimental energy distribution of α - particles, the effective radii R_{A}^{neck} of the neck of the compound fissile nucleus were calculated using the formula (5) in [S. G.

Kadmensky, L. V. Titova, D. E. Lubashevsky, Phys. At. Nucl., 83, 581 (2020)] for different parent nuclei. These radii have the values $R_A^{\mathrm{neck}} \approx (2-3)$ Fm, which are significantly smaller than values of parent nuclei radii and are reasonably corresponded to the values of neck radii of parent nuclei found in articles [O. Serot et al., Eur. Phys. J., A8 187 (2000); S. Vermote et al., Nucl. Phys. A 806 (2008)] in which the neck of a fissionable nucleus is estimated in the calculations of the evolution of its shape as it moves towards to the scission point on the basis of a generalized drop model of the nucleus.

Table 5. Characteristics of spontaneous ternary fission of ^{248}Cm, ^{250}Cf and ^{252}Cf nuclei.

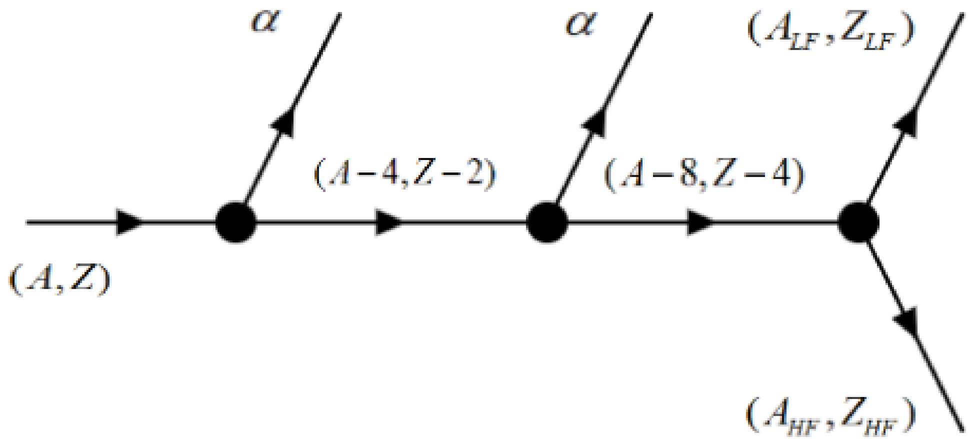
Nucleus	^{248}Cm	^{250}Cf	^{252}Cf
Q_α^A, MeV	5,16	6,13	6,22
$(T_\alpha^A)_0$, MeV	14,72	15,95	15,96
$\bar{\Gamma}_\alpha^A$, MeV	$3{,}81 \cdot 10^{-35}$	$1{,}11 \cdot 10^{-30}$	$5{,}31 \cdot 10^{-30}$
$\bar{\Gamma}_f^A$, MeV	$3{,}48 \cdot 10^{-36}$	$8{,}84 \cdot 10^{-34}$	$1{,}67 \cdot 10^{-31}$
$\bar{\Gamma}_\alpha^{A-4}$, MeV	$1{,}80 \cdot 10^{-37}$	$3{,}03 \cdot 10^{-33}$	$3{,}81 \cdot 10^{-35}$

$\bar{\Gamma}_f^{A-4}$, MeV	$2{,}17 \cdot 10^{-40}$	$9{,}11 \cdot 10^{-37}$	$3{,}48 \cdot 10^{-36}$
$\bar{\Gamma}_{\alpha f}^A / \bar{\Gamma}_f^A$	$(2{,}44 \pm 0{,}11) \cdot 10^{-3}$	$(3{,}87 \pm 0{,}30) \cdot 10^{-3}$	$(3{,}87 \pm 0{,}30) \cdot 10^{-3}$
$T_{\alpha,m}^A$, MeV	17,2	19,0	18,9
A_m , MeV	0,030	0,045	0,045
R_A^{neck}, Fm	3,19	2,19	2,18

6. QUATERNARY FISSION OF NUCLEI AS THREE-STAGE VIRTUAL PROCESS

New approach to describing the quaternary fission characteristics using the theory of quaternary fission as a virtual process developed in [Kadmensky, S.G., Titova, L.V., and Lyubashevsky, D.E., Phys. At. Nucl., **83**, 581 (2020)] was proposed in article [Kadmensky, S.G., Titova, Bull. Russ. Acad. Sci., 85, 569 (2021)]. The quaternary fission amplitude is in this case represented by the Feynman diagram where horizontal arrows

Fig. 8



correspond to the Green functions of intermediate nuclei, so this quaternary fission is a three-step virtual process [Kadmensky, S.G. and Titova, L.V., Phys. At. Nucl., **73**, 16 (2013)].

The widths of the quaternary fission process has form:

$$\Gamma_{\alpha \mathrm{f}}^{A} = \frac{1}{(2\pi)^2} \iint \frac{\Gamma_{\alpha_1}^{A}\left(T_{\alpha_1}\right) \Gamma_{\alpha_2}^{A-4}\left(T_{\alpha_2}\right) \Gamma_{\mathrm{f}}^{A-8}\left(Q_{\mathrm{f}} - T_{\alpha_1} - T_{\alpha_2}\right)}{\left(Q_{\alpha_1}^{A} - T_{\alpha_1}\right)^2 \left(Q_{\alpha_2}^{A-4} - T_{\alpha_2}\right)^2} dT_{\alpha_1} dT_{\alpha_2} \quad (7),$$

where $\Gamma_{\alpha_1}^{A}\left(T_{\alpha_1}\right)$ and $\Gamma_{\alpha_2}^{A-4}\left(T_{\alpha_2}\right)$ are the widths of the α-decay of the ground states of the parent $\left(A, Z\right)$ and intermediate $\left(A-4, Z-2\right)$ nuclei with the emission of α-particles having kinetic energies T_{α_1} and T_{α_2} from the neck and formation of the ground state of the daughter nucleus $\left(A-8, Z-4\right)$, which corresponds to configurations (0), $\Gamma_{\mathrm{f}}^{A-8}$ is the fission width if the daughter nucleus $\left(A-8, Z-4\right)$, which corresponds to configuration (0), and Q_{f} is the heat of quaternary fission of parent nucleus $\left(A, Z\right)$. Further analysis will use experimental data from Tables 5 and 6.

Table 6. Characteristics of the quaternary fission of nuclei ^{248}Cm,^{252}Cf and compound nuclei ^{234}U,^{236}U

Nucleus	^{248}Cm	^{252}Cf	^{234}U	^{236}U
T_{α_1}, MeV	14.3	15.9	15.7	15.5
T_{α_2}, MeV	10.1	12.7	11.3	10.7
$Q^A_{\alpha_1}$, MeV	5.16	6.21	4.85	4.55
$Q^A_{\alpha_2}$, MeV	4.66	5.16	4.77	4.08
B_n, MeV	-	-	6.85	6.55
$N_{\alpha\alpha} \cdot 10^7$	(1.4±0.3)	(9.72±3.26)	(0.89±0.28)	(0.54±0.17)
$N_{\alpha_1} \cdot 10^3$	(2.3±0.3)	(3.24±0.12)	(2.17±0.07)	(1.70±0.03)

$N_{\alpha_2} \cdot 10^5$	(3.04±0.24)	(15.0±5.0)	(2.05±0.65)	(1.6±0.5)

$\left(W_{\alpha}^{A}\right)_{max}$, МэВ	12.2	13.7	12.9	13.3
$FWHM_{\alpha}^{A}$, MeV	10.5	11.3	10.9	10.9
$\left(W_{\alpha}^{A}\right)_{max}$, МэВ	14.3	15.9	15.7	15.5
$FWHM_{\alpha}^{A}$, MeV	11.3	9.8	9.8	9.8
$\left(W_{\alpha_2}^{A}\right)_{max}$, MeV	10.1	12.7	11.3	10.7
$FWHM_{\alpha_2}^{A}$, MeV	9.8	8.6	8.2	9.3
$\left(T_{\alpha_1}\right)_{max}$, MeV	15.9	18.7	20.2	20.0

$\left(T_{\alpha}^{A}\right)_{max}$, MeV	12.7	14.9	14.1	14.2
$P_{\alpha_2}/P_{\alpha_1}$	0.006	0.025	0.026	0.025

From the ratio between quaternary fission width $\Gamma_{\alpha f}^{A}$ and spontaneous binary fission width Γ_{f}^{A} we can obtain the energy distribution $W_{\alpha\alpha}$ of emitted α-particles, normalized to their yield $N_{\alpha\alpha}$ in quaternary fission:

$$W_{\alpha\alpha} = \frac{\Gamma_{\alpha_1}^{A}\left(T_{\alpha_1}\right)\Gamma_{\alpha_2}^{A-4}\left(T_{\alpha_2}\right)\Gamma_{f}^{A-8}\left(Q_f - T_{\alpha_1} - T_{\alpha_2}\right)}{N_{\alpha\alpha}\left(Q_{\alpha_1}^{A} - T_{\alpha_1}\right)^2\left(Q_{\alpha_2}^{A-4} - T_{\alpha_2}\right)^2\Gamma_{f}^{A}} \qquad (8).$$

Taking into account that widths $\Gamma_{\mathrm{f}}^{A-8}$ is close to width Γ_{f}^{A} because of the sub-barier character of the fission of nuclei $(A-8, Z-4)$ and (A, Z), separating the energy distributions of the first $W_{\alpha_1}(T_{\alpha_1})$ and the second $W_{\alpha_2}(T_{\alpha_2})$ in energy distribution $W_{\alpha\alpha}$, using yields of the first and the second α-particles from Table, according to the consequent character of quaternary fission, taking into account that the neck radii $R_{\mathrm{neck}}^{A} \approx R_{\mathrm{neck}}^{A-4}$, the ratio of the probabilities of the first and second α particles penetrating through the Coulomb barrier was constructed

for spontaneous fission:

$$\frac{P\left(T_{\alpha_2}\right)}{P\left(T_{\alpha_1}\right)}=\frac{\sqrt{\left(T_{\alpha_2}\right)_{\max}}N_{\alpha_2}\left(W_{\alpha_2}\left(T_{\alpha_2}\right)\left(Q_{\alpha_2}^{A-4}-T_{\alpha_2}\right)^2\right)_{\max}}{\sqrt{\left(T_{\alpha_1}\right)_{\max}}N_{\alpha_1}\left(W_{\alpha_1}\left(T_{\alpha_1}\right)\left(Q_{\alpha_1}^{A}-T_{\alpha_1}\right)^2\right)_{\max}} \qquad (9)$$

for fission induced by neutrons:

$$\frac{P\left(T_{\alpha_2}\right)}{P\left(T_{\alpha_1}\right)}=\frac{\sqrt{\left(T_{\alpha_2}\right)_{\max}}N_{\alpha_2}\left(W_{\alpha_2}\left(T_{\alpha_2}\right)\left(Q_{\alpha_2}^{A-4}-T_{\alpha_2}\right)^2\right)_{\max}}{\sqrt{\left(T_{\alpha_1}\right)_{\max}}N_{\alpha_1}\left(W_{\alpha_1}\left(T_{\alpha_1}\right)\left(Q_{\alpha_1}^{A}+\left|B_{\mathrm{n}}\right|-T_{\alpha_1}\right)^2\right)_{\max}} \qquad (10)$$

where experimental energy distribution of the first and the second emitted α-particles:

$$W_{\alpha_i}\left(T_{\alpha_i}\right)=\frac{1}{\sqrt{2\pi}\sigma_{\alpha_i}}\exp\left(\frac{T_{\alpha_i}-\left\langle T_{\alpha_i}\right\rangle}{2\sigma_{\alpha_i}^2}\right) \qquad (11)$$

The resulting ratios are 0.006 and 0.025 for spontaneous fission of ^{248}Cm and ^{252}Cf nuclei, respectively, and 0.026 and 0.025 for the neutron-induced fission of the uranium target nuclei ^{233}U and ^{235}U nucleus. This values are in agreement with results [Kadmensky, S.G. and Titova, L.V., Phys. At. Nucl., 2013, vol. 73, no. 1, p. 16.]

The barrier penetrability ratios for the first and second α particles confirm that the probability of the second α particle being emitted is about 10^3 times lower.

The estimations of kinetic energies and of the α particles at which widths for the first and second alpha-particles have maximum are achieved. These correspond to the effective Coulomb barrier heights for the first and second α particles emitted from the neck of the fissioning nucleus, which are estimated to be 15.9 and 12.7 MeV for the fission of the ^{248}Cm nucleus and 18.7 and 14.9 MeV for the fission of the ^{252}Cf nucleus.

7. CONCLUSION

The submitted work demonstrates the positive consequences of the introduction of virtual states of different intermediate nuclei for the description of two-stage and three-stage nuclear decays. This allows to highlight the new class of nuclear decays with virtual characteristics including presently $2p$ - and 2β - decays, as well as ternary and quaternary fission of nuclei. It would be interesting to continue the expansion of this class by introducing in it new virtual sequential decays. The number of such decays can probably be attributed to the emission of prescission neutrons by fissile nuclei.