LOW-ENERGY SCATTERING OF A POLARIZED NEUTRON ON A POLARIZED PROTON

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The spin structure of the amplitude of s -wave elastic scattering of a slow neutron on a free proton is considered. The formula for effective cross-section of scattering of a slow polarized neutron on the polarized proton target is obtained, and it is shown that, in the effective cross-section, the amplitudes of triplet (total spin $S = 1$) and singlet (total spin $S = 0$) scattering are summed up incoherently. In doing so, the maximum value of integral cross-section $\sigma_{max} \approx 37.1$ barn corresponds to the antiparallel orientations of spins of the totally polarized neutron and proton ($\mathbf{P}_n \mathbf{P}_p = -1$, where $\mathbf{P}_n$ and $\mathbf{P}_p$ are the polarization vectors), whereas the minimal magnitude of cross-section $\sigma_{min} \approx 3.64$ barn is observed in the case of parallel orientation of spins ($\mathbf{P}_n \mathbf{P}_p = +1$) .

The low-energy scattering of polarized neutrons is also analyzed . Taking into account the identity effect, the s -wave scattering of neutrons is possible only in the singlet state. According to the isotopic invariance, one could expect that the amplitude of s -wave scattering of two neutrons must coincide, with the precision of $(1 \div 3)$ %, with the singlet amplitude of neutron scattering on the proton . Nevertheless, in fact the scattering lengths take the values $b^{(np)} = -a^{(np)}(0) = -23.7$ fm and $b^{(nn)} = -a^{(nn)}(0) = -17.0$ fm ; so, we have $| b^{(np)} / b^{(nn)} | \approx 1.4$. However, the concept of "isotopic invariance" is applicable rather to the interaction potential than to the scattering amplitude. Analysis shows that, if the modulus of amplitude is large as compared with the range of force action, the magnitude of scattering amplitude becomes very sensitive to the parameters of potential. Within the model of spherical rectangular well, the change of depth or width of the well by several per cent leads to the substantial change of scattering length .

1. Amplitude of the elastic s-wave scattering of a slow neutron on a free proton has the following structure:

$$\hat{f}(k) = a_t(k)\hat{P}_t + a_s(k)\hat{P}_s \quad , \tag{1}$$

where $a_t(k)$ is the amplitude of triplet scattering, corresponding to the total spin of the neutron-proton system equaling 1, $a_s(k)$ is the amplitude of singlet scattering, corresponding to the total spin of the (np) system equaling 0, k is the modulus of nucleon momentum in the c.m. frame ,

$$\hat{P}_t = \frac{3\hat{I}^{(1)} \otimes \hat{I}^{(2)} + \hat{\boldsymbol{\sigma}}^{(1)} \otimes \hat{\boldsymbol{\sigma}}^{(2)}}{4} \quad , \qquad \hat{P}_s = \frac{\hat{I}^{(1)} \otimes \hat{I}^{(2)} - \hat{\boldsymbol{\sigma}}^{(1)} \otimes \hat{\boldsymbol{\sigma}}^{(2)}}{4} \tag{2}$$

are the operators of projection onto the triplet states and the singlet state, respectively, satisfying the relations:

$$\hat{P}_t^{\,2} = \hat{P}_t \quad , \quad \hat{P}_s^{\,2} = \hat{P}_s \quad , \quad \hat{P}_s \hat{P}_t = \hat{P}_t \hat{P}_s = 0 \tag{3}$$

(In Eq. (2): $\hat{I}^{(1)}$, $\hat{I}^{(2)}$ are two-row unit matrices, $\hat{\boldsymbol{\sigma}}^{(1)} = \{ \hat{\sigma}_x^{(1)}, \hat{\sigma}_y^{(1)}, \hat{\sigma}_z^{(1)} \}$ and $\hat{\boldsymbol{\sigma}}^{(2)} = \{ \hat{\sigma}_x^{(2)}, \hat{\sigma}_y^{(2)}, \hat{\sigma}_z^{(2)} \}$ are vector Pauli operators) .

Relations (3) can be easily verified using the matrix equality:

$$(\hat{\boldsymbol{\sigma}}^{(1)} \otimes \hat{\boldsymbol{\sigma}}^{(2)})^2 = 3\hat{I}^{(1)} \otimes \hat{I}^{(2)} - 2\hat{\boldsymbol{\sigma}}^{(1)} \otimes \hat{\boldsymbol{\sigma}}^{(2)} \quad . \tag{4}$$

Formula (1) may be also rewritten in the form:

$$\hat{f}(k) = c(k)\hat{I}^{(1)} \otimes \hat{I}^{(2)} + d(k)\hat{\boldsymbol{\sigma}}^{(1)} \otimes \hat{\boldsymbol{\sigma}}^{(2)} \quad , \tag{5}$$

where

$$c(k) = \frac{3a_t(k) + a_s(k)}{4} \quad , \quad d(k) = \frac{a_t(k) - a_s(k)}{4} \quad .$$

According to the experimental data, at zero energy:

$a_t(0)$ = –5.38 fm ; $a_s(0)$ = + 23.6 fm; c (0) = 1.89 fm; d (0) = –7.27 fm .

By definition, the scattering lengths are: $b_{(s,\,t)} = -\, a_{(s,\,t)}(0)$.

Introducing the operator of permutation of spin projections [1,2] :

$$\hat{P}^{(1,2)}_{exch} = \frac{1}{2}[\hat{I}^{(1)} \otimes \hat{I}^{(2)} + \hat{\boldsymbol{\sigma}}^{(1)} \otimes \hat{\boldsymbol{\sigma}}^{(2)}] \quad , \tag{6}$$

one can get easily convinced in the validity of the following equalities as well :

$$\hat{P}^{(1,2)}_{exch} \hat{P}_t = \hat{P}_t \hat{P}^{(1,2)}_{exch} = +\hat{P}_t \quad , \quad \hat{P}^{(1,2)}_{exch} \hat{P}_s = \hat{P}_s \hat{P}^{(1,2)}_{exch} = -\hat{P}_s \quad . \tag{7}$$

In accordance with Eq. (7), the eigenvalues of the operator $\hat{P}_{exch}^{(1,2)}$ are equal to (+1) for the three triplet states and to (–1) for the singlet state .

2. The total cross-section of elastic scattering of slow polarized neutrons on a polarized proton target amounts to:

$$\sigma^{(n,p)} = 4\pi\, tr(\hat{f}(k)\hat{\rho}^{(1,2)}\hat{f}^{+}(k)) = 4\pi\, tr(\hat{f}(k)\hat{f}^{+}(k)\hat{\rho}^{(1,2)}) \quad , \qquad (8)$$

where the amplitude $\hat{f}(k)$ is determined according to Eq. (1) ,

$$\hat{\rho}^{(1,2)} = \frac{1}{4}(\hat{I}^{(1)} + \mathbf{P}_n\hat{\boldsymbol{\sigma}}^{(1)}) \otimes (\hat{I}^{(2)} + \mathbf{P}_p\hat{\boldsymbol{\sigma}}^{(2)}) \qquad (9)$$

is the 4-row spin density matrix of the neutron and proton,
$\mathbf{P}_n = \langle\hat{\boldsymbol{\sigma}}^{(1)}\rangle = tr(\hat{\boldsymbol{\sigma}}^{(1)}\hat{\rho}^{(1,2)})$ and $\mathbf{P}_p = \langle\hat{\boldsymbol{\sigma}}^{(2)}\rangle = tr(\hat{\boldsymbol{\sigma}}^{(2)}\hat{\rho}^{(1,2)})$ are respective polarization vectors for the neutron and proton .

Taking into account the equalities (2), we have :

$$tr(\hat{P}_t^{\,2}\hat{\rho}^{(1,2)}) = tr(\hat{P}_t\,\hat{\rho}^{(1,2)}) = \frac{3+\mathbf{P}_n\mathbf{P}_p}{4} \quad , \quad tr(\hat{P}_s^{\,2}\hat{\rho}^{(1,2)}) = tr(\hat{P}_s\,\hat{\rho}^{(1,2)}) = \frac{1-\mathbf{P}_n\mathbf{P}_p}{4} \quad ,$$

$$tr(\hat{P}_t\hat{P}_s\,\hat{\rho}^{(1,2)}) = 0 \quad . \qquad (10)$$

As a result, we obtain:

$$\sigma^{(np)} = 4\pi \left\{ |a_t(k)|^2 \frac{3+\mathbf{P}_n\mathbf{P}_p}{4} + |a_s(k)|^2 \frac{1-\mathbf{P}_n\mathbf{P}_p}{4} \right\} =$$

$$= 4\pi \left\{ \frac{3|a_t(k)|^2 + |a_s(k)|^2}{4} + \frac{|a_t(k)|^2 - |a_s(k)|^2}{4}(\mathbf{P}_n\mathbf{P}_p) \right\} \quad . \qquad (11)$$

We see that, in the effective cross-section, the contributions of triplet and singlet scattering are summed up incoherently. In doing so, the quantities

$$W_t = \frac{3+\mathbf{P}_n\mathbf{P}_p}{4} \quad , \quad W_s = \frac{1-\mathbf{P}_n\mathbf{P}_p}{4} \qquad (12)$$

determine the respective relative fractions of the triplet states and the singlet state for the (np) system with the polarization vectors $\mathbf{P}_n$ and $\mathbf{P}_p$.

In the case when at least one of the nucleons is unpolarized, as well as when the nucleon polarization vectors are mutually orthogonal ($\mathbf{P}_n \mathbf{P}_p = 0$), the integral cross-section of s-wave elastic neutron scattering on the proton at zero energy equals:

$$\sigma_0 = \pi \, (\, 3a_t^{\,2}(0) + a_s^{\,2}(0) \,) \approx 20.4 \ \text{barn} \, .$$

The maximum value of the effective cross-section corresponds to the antiparallel orientation of spins of the totally polarized proton and neutron ($\mathbf{P}_n \mathbf{P}_p = -1$) :

$$\sigma_{max} = 2\pi \, (\, a_t^{\,2}(0) + a_s^{\,2}(0) \,) \approx 37.1 \ \text{barn} \, ,$$

and the minimal value of the effective cross-section – to the parallel orientation of spins ($\mathbf{P}_n \mathbf{P}_p = +1$) :

$$\sigma_{min} = 4\pi \, a_t^{\,2}(0) \approx 3.64 \ \text{barn} \, .$$

3. Let us remark that the amplitude of s -wave scattering of slow neutrons, defined in the front hemisphere of solid angles, is as follows, taking into account their identity [3] :

$$f^{(nn)}(k) = a_s^{(nn)}(k)\hat{P}_s - a_s^{(nn)}(-k)\hat{P}_{exch}^{(1,2)}\hat{P}_s = 2a_s^{(nn)}(k)\hat{P}_s =$$

$$= a_s^{(nn)}(k) \, \frac{\hat{I}^{(1)} \otimes \hat{I}^{(2)} - \hat{\sigma}^{(1)} \otimes \hat{\sigma}^{(2)}}{2} \quad . \tag{13}$$

In doing so, the integral cross-section of elastic scattering equals:

$$\sigma^{(nn)} = 2\pi \, | \, a_s^{(nn)}(k) \, |^2 \, (1 - \mathbf{P}_1 \, \mathbf{P}_2) \quad , \tag{14}$$

where $\mathbf{P}_1$, $\mathbf{P}_2$ – neutron polarization vectors .

Indeed, owing to the equality [2,3] :

$$(-1)^{L+S} = 1 \quad , \tag{15}$$

where L is the orbital momentum and S is the total spin of two identical particles, the *s*-wave neutron-neutron scattering – as well as the s -wave scattering of any two identical particles with spin 1/2 – is possible only in the singlet state .

Taking into account the isotopic invariance, one could anticipate that the amplitude of neutron scattering on the proton, corresponding to the total spin of the (np) system $S = 0$ (singlet amplitude $a_s^{(np)}(k)$) must coincide with the neutron-neutron scattering amplitude $a_s^{(nn)}(k)$, which can be singlet only . However, in fact these amplitudes are essentially different: the singlet length of (np) scattering is $b_s^{(np)} = -23.7$ fm, whereas the respective length of (nn) scattering equals $b_s^{(nn)} = -17.0$ fm. This is due to the fact that, under the condition when the scattering amplitude substantially exceeds in modulus the range of force action, the scattering length becomes very sensitive to the depth and width of the interaction potential [4]. In this case, the concept of isotopic invariance is applicable namely to the interaction potential, but not to the scattering amplitude: the potentials of neutron-proton and neutron-neutron interaction should be approximately the same, differing by not more than (2 ÷ 3) % . Nevertheless, this small difference leads to the difference of respective amplitudes by several tens per cent, which looks like a spurious violation of isotopic invariance .

4. The further analysis will be performed within the model of a rectangular well with depth V_0 and width r_0 . In this model, the phase of s -wave scattering $\delta_0(k)$ is determined by means of the conjunction of logarithmic derivatives of wave functions inside and outside the well [3] . This gives:

$$k \operatorname{ctg}(kr_0 + \delta_0(k)) = \chi(k) \operatorname{ctg} \chi(k) \quad , \tag{16}$$

where

$$\chi(k) = \sqrt{\frac{2mV_0}{\hbar^2} + k^2} \quad , \tag{17}$$

m is the reduced mass.

In doing so,

$$\mathrm{ctg}(kr_0 + \delta_0(k)) = \frac{\mathrm{ctg}\,\delta_0(k) - \mathrm{tg}\,kr_0}{1 + \mathrm{ctg}\,\delta_0(k)\,\mathrm{tg}\,kr_0} \quad . \tag{18}$$

As it is known, the amplitude of s -wave scattering is connected with the phase $\delta_0(k)$ by the relation:

$$\frac{1}{a(k)} = k\,\mathrm{ctg}\,\delta_0(k) - ik \quad . \tag{19}$$

Substituting Eqs. (18) and (19) into formula (16), we obtain:

$$\frac{1}{a(k)} = \frac{\chi(k)\,\mathrm{ctg}\,(\chi(k)\,r_0) + k\,\mathrm{tg}\,kr_0}{1 - \frac{\mathrm{tg}\,kr_0}{k}\,\chi(k)\,\mathrm{ctg}\,(\chi(k)\,r_0)} - ik \quad . \tag{20}$$

Hence, the scattering length is

$$b = -a(0) = -\frac{\mathrm{tg}\,(\chi_0\,r_0)}{\chi_0} + r_0 \quad , \tag{21}$$

where

$$\chi_0 = \chi(0) = \sqrt{\frac{2mV_0}{\hbar^2}} \quad . \tag{22}$$

In the framework of the effective radius theory [3] :

$$\frac{1}{a(k)} = -\frac{1}{b} + \frac{1}{2} r_{eff}\, k^2 - ik \quad . \tag{23}$$

Expanding the right-hand side of Eq. (20) over the powers of k^2 and taking into account that $\chi(k) = \chi_0 + k^2/2\chi_0$, we obtain formula (21) for the scattering length and the following value of the effective radius:

$$r_{eff} = 2\, r_0\, (1-\chi_0 r_0\, \mathrm{ctg}\,(\chi_0 r_0))^{-1} \;\; +$$

$$+ \left[\frac{\mathrm{ctg}(\chi_0 r_0)}{\chi_0} - \frac{r_0}{\sin^2(\chi_0 r_0)} + \frac{2}{3} r_0^{\,3} \chi_0^{\,2}\, \mathrm{ctg}^2(\chi_0 r_0)\right] (1-\chi_0 r_0\, \mathrm{ctg}(\chi_0 r_0))^{-2} \,. \qquad (24)$$

It follows from Eq. (21) for the scattering length that at $\chi_0 r_0 \approx \pi/2$ $(\mathrm{tg}\,(\chi_0 r_0) >> 1)$ the values of scattering length are very sensitive to the magnitudes of depth and width of the potential well .

Indeed, let us introduce the characteristics of the potential well for the proton and neutron in the singlet state, corresponding to the data on the amplitude of singlet (np) scattering:

$$V_{0(s)}^{(np)} = 14.524 \;\text{fm} \;\; ; \qquad r_{0(s)}^{(np)} = 2.5458 \;\text{fm} \;\; .$$

In this case ($\chi_0 r_0 = 1.506$, $\mathrm{tg}\,(\chi_0 r_0) = 15.47 >> 1$), calculations using Eqs. (21) , (24) give:

$$b_s^{(np)} = -23.6 \;\text{fm} \;, \qquad r_{eff\,(s)}^{(np)} = 2.66 \;\text{fm} \;.$$

When decreasing the depth of the potential well by 3 % only ($V_0^{(nn)} = 14.084$ MeV) , the calculation according to Eqs. (21), (24) at the same width of the potential well leads to the values :

$$b_s^{(nn)} = -17.1 \;\text{fm} \;, \qquad r_{eff\,(s)}^{(nn)} = 2.72 \;\text{fm} \quad .$$

In doing so, $\left| b_s^{(np)} \big/ b_s^{(nn)} \right| = 1.38$, whereas the effective radius changes insignificantly.

Now let the depth of the potential well take the initial value ($V_0^{(nn)} = 14.524$ fm), and let us decrease the width of the well by 1.5 % ($r_{0(s)}^{(nn)} = 2.49$ fm) . Then the calculation using Eq. (21) leads to the value $b_s^{(nn)} = -16.8$ fm .

So, we see that, despite the substantial difference of the singlet amplitudes of (*np*) and (*nn*) scattering, for the respective potentials the isotopic invariance really holds: the parameters of potentials coincide with the precision of $(1.5 \div 3)\ \%$.

5. Summary

1. The spin structure of the amplitude and effective cross-section of scattering of a slow neutron on a proton target is analyzed .

2. The comparison of the (*np*) scattering amplitude with the neutron-neutron scattering one is performed .

3. It is shown that, in the conditions when the modulus of scattering length consi-derably exceeds the range of interaction, the value of scattering length becomes very sensitive to the characteristics of potential .

4. In this connection, the concept of isotopic invariance is related rather to two-nucleon interaction potentials than to nucleon-nucleon scattering amplitudes .

References

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