LOW-ENERGY ELASTIC SCATTERING OF A POLARIZED NEUTRON ON A POLARIZED DEUTERON

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Abstract

The spin structure of the amplitude of *S*-wave elastic scattering of slow neutrons on deuterons is analyzed . The operators of projection onto the quartet state (total spin J = 3/2) and doublet state (total spin J = 1/2) of the (*nd*) system and the scattering amplitudes, corresponding to these states, are introduced. It is shown that in the low-energy limit the dependence of the effective cross section of (*nd*)-scattering upon the neutron and deuteron polarization vectors is determined by only two scattering amplitudes (the quartet and doublet ones), the contributions of which are summed up incoherently. The expression for the correlation tensor of the final (*nd*) system at low-energy scattering of unpolarized neutrons on unpolarized deuterons is derived as well .

1. *S*-wave scattering on a nucleus

The amplitude of *S*-wave scattering of a hadron c on a nucleus d has the following spin structure :

$$\hat{f} = a\,(\hat{I}^{(c)} \otimes \hat{I}^{(d)}) + b\,(\hat{\mathbf{s}}_c \otimes \hat{\mathbf{s}}_d) \quad , \qquad (1)$$

where $\hat{I}^{(c)}$ and $\hat{I}^{(d)}$ are unit matrices, $\hat{\mathbf{s}}_c$ is the hadron spin operator and $\hat{\mathbf{s}}_d$ is the nucleus spin operator .

Now let us pass to the representation of states of the two-particle system (*cd*) with the definite total spins. According to the momentum addition law, the total spin of the (*cd*) system may take integer or half-integer values in the range :

$$| s_c - s_d | \le J \le s_c + s_d \quad , \qquad (2)$$

where s_c and s_d are respective spins of the hadron and the nucleus . We will consider the *S*-wave scattering at zero energy, and let us introduce the lengths $a_J = - f_J$ corresponding to the total spins J :

$$a_J = -a - b \langle \hat{\mathbf{s}}_c \otimes \hat{\mathbf{s}}_d \rangle_J \quad , \qquad (3)$$

where the scalar products of spin operators in the states with the definite values of J are determined in accordance with the well-known formula from the theory of vector addition of angular momenta :

$$\langle \hat{\mathbf{s}}_c \otimes \hat{\mathbf{s}}_d \rangle_J = \frac{J(J+1) - s_c(s_c+1) - s_d(s_d+1)}{2} \quad . \qquad (4)$$

Since the trace of a matrix does not change under unitary transformations from one representation to another, the following relation holds :

$$tr(\hat{\mathbf{s}}_c \otimes \hat{\mathbf{s}}_d) = \sum_{J=|s_c-s_d|}^{s_c+s_d} \langle \hat{\mathbf{s}}_c \otimes \hat{\mathbf{s}}_d \rangle_J (2J+1) \quad . \qquad (5)$$

The left-hand side of Eq. (5) contains the matrix trace in the standard representation of two one-particle spin states with the definite projections, whereas the right-hand side of Eq. (5) contains the matrix trace in the representation of states of the two-particle system (hadron – nucleus) with the definite values of total spin J, the number of total spin projections equaling $(2J+1)$.

In the case of unpolarized hadron c or nucleus d , we have :

$$tr(\hat{\mathbf{s}}_c \otimes \hat{\mathbf{s}}_d) = 0 \quad .$$

Then, taking into account Eq. (5), we obtain :

$$\sum_{J=|s_c-s_d|}^{s_c+s_d} (2J+1) \langle \hat{\mathbf{s}}_c \otimes \hat{\mathbf{s}}_d \rangle_J = 0 \quad . \qquad (6)$$

Multiplying the left-hand and right-hand sides of the equality (3) by $(2J+1)$ and performing the summation over total spins J in the range given by (2), we come, with using Eq. (6), to the relation :

$$\sum_J (2J+1) a_J = -a \sum_J (2J+1) \quad . \qquad (7)$$

Since unitary transformations do not change the number of basis states, we may write :

$$\sum_{J=|s_c-s_d|}^{s_c+s_d}(2J+1) = (2s_c+1)(2s_d+1) \quad . \tag{8}$$

So, it follows from here that :

$$a = -\frac{\sum_{J=|s_c-s_d|}^{s_c+s_d}(2J+1)\ a_J}{(2s_c+1)(2s_d+1)} \quad . \tag{9}$$

2. Coherent length of scattering

The coherent length of scattering of the hadron c on the unpolarized nucleus d has the same absolute value and opposite sign as compared with the amplitude of "forward" scattering at zero energy in the laboratory frame, averaged over all the spin states of hadron c and nucleus d . Under the transition from the c.m. frame to the laboratory frame at zero angle, we find :

$$A_{coh} = -\frac{m_d+m_c}{m_d}\ tr(\hat{f})\ \frac{1}{(2s_c+1)(2s_d+1)} \quad , \tag{10}$$

where m_d and m_c are masses of the nucleus d and hadron c, respectively.

On account of Eq. (5), we obtain by using formula (1) :

$$A_{coh} = -\frac{m_d+m_c}{m_d}\ a \quad , \tag{11}$$

or, according to Eq. (9) ,

$$A_{coh} = \frac{m_d+m_c}{m_d}\ \frac{\sum_{J=|s_c-s_d|}^{s_c+s_d}(2J+1)\ a_J}{(2s_c+1)(2s_d+1)} \quad . \tag{12}$$

3. *S*-wave neutron scattering on a deuteron .

Now let us pass to the low-energy neutron scattering on the deuteron. The total spin of the (*nd*) system takes two values: $J = 3/2$ and $J = 1/2$. Instead of the neutron spin operator $\hat{\mathbf{s}}_n$, it is convenient to use the Pauli operator $\hat{\boldsymbol{\sigma}}_n = 2\,\hat{\mathbf{s}}_n$.

According to Eq. (4), we have :

$$\langle\, \hat{\boldsymbol{\sigma}}_n \otimes \hat{\mathbf{s}}_d \,\rangle_{3/2} = 1 \quad , \quad \langle\, \hat{\boldsymbol{\sigma}}_n \otimes \hat{\mathbf{s}}_d \,\rangle_{1/2} = -2 \quad . \tag{13}$$

Here $\hat{\mathbf{s}}_d$ is the vector operator of deuteron spin .

The operator of projection onto the states with total spin $J = 3/2$ in the 6-dimensional spin space of the neutron–deuteron system takes the form :

$$\hat{P}_{3/2} = \frac{2\,\hat{I}_{nd} + (\,\hat{\boldsymbol{\sigma}}_n \otimes \hat{\mathbf{s}}_d\,)}{3} \quad , \tag{14}$$

where $\hat{I}_{nd} = (\,\hat{I}_n \otimes \hat{I}_d\,)$ is the unit matrix .

In doing so, four eigenvalues of the operator $\hat{P}_{3/2}$, corresponding to the total spin $J = 3/2$, are equal to 1, whereas two eigenvalues corresponding to the total spin $J = 1/2$ are equal to zero .

Meantime, the operator of projection onto the states with total spin $J = 1/2$ in the 6-dimensional spin space of the (*nd*) system is as follows :

$$\hat{P}_{1/2} = \frac{\hat{I}_{nd} - (\,\hat{\boldsymbol{\sigma}}_n \otimes \hat{\mathbf{s}}_d\,)}{3} \quad , \tag{15}$$

here two eigenvalues of the operator $\hat{P}_{1/2}$, corresponding to the total spin $J = 1/2$, are equal to 1, and four eigenvalues corresponding to the total spin $J = 3/2$ are equal to zero .

It is easy to see that $\hat{P}_{3/2} = \hat{P}^{+}_{3/2}$, $\hat{P}_{1/2} = \hat{P}^{+}_{1/2}$ and

$$\hat{P}_{3/2} + \hat{P}_{1/2} = \hat{I}_{nd} \quad . \tag{16}$$

In doing so,

$$\hat{P}_{3/2}{}^2 = \hat{P}_{3/2} \;, \quad \hat{P}_{1/2}{}^2 = \hat{P}_{1/2} \;, \quad \hat{P}_{1/2}\,\hat{P}_{3/2} = \hat{P}_{3/2}\,\hat{P}_{1/2} = 0 \quad . \tag{17}$$

The amplitude of S-wave neutron scattering on the deuteron in the c.m. frame at energies being close to zero can be presented as a linear combination of the quartet ($a_{3/2}$) and doublet ($a_{1/2}$) scattering lengths :

$$\begin{aligned} \hat{f} &= -(\,a_{3/2}\,\hat{P}_{3/2} + a_{1/2}\,\hat{P}_{1/2}\,) = \\ &= -\left(\frac{2\,a_{3/2} + a_{1/2}}{3}\right)\hat{I}_{nd} + \left(\frac{a_{3/2} - a_{1/2}}{3}\right)(\,\hat{\boldsymbol{\sigma}}_n \otimes \hat{\mathbf{s}}_d\,) \quad . \end{aligned} \tag{18}$$

Taking into account the relations (14) and (15) , the S-wave differential cross section of neutron scattering on the deuteron at energies being close to zero is described by the formula :

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= tr\,(\,\hat{f}\,\hat{\rho}_{nd}\,\hat{f}^{+}) = |\,a_{3/2}\,|^2\,tr\,(\,\hat{P}_{3/2}\,\hat{\rho}_{nd}\,\hat{P}_{3/2}) + |\,a_{1/2}\,|^2\,tr\,(\,\hat{P}_{1/2}\,\hat{\rho}_{nd}\,\hat{P}_{1/2}) + \\ &+ a_{3/2}\,a^{*}_{1/2}\,tr\,(\,\hat{P}_{3/2}\,\hat{\rho}_{nd}\,\hat{P}_{1/2}) + a^{*}_{3/2}\,a_{1/2}\,tr\,(\,\hat{P}_{1/2}\,\hat{\rho}_{nd}\,\hat{P}_{3/2}) = \\ &= |\,a_{3/2}\,|^2\,tr\,(\,\hat{P}_{3/2}\,\hat{\rho}_{nd}) + |\,a_{1/2}\,|^2\,tr\,(\,\hat{P}_{1/2}\,\hat{\rho}_{nd}) \quad . \end{aligned} \tag{19}$$

Here $\hat{\rho}_{nd}$ is the primary 6-dimensional spin density matrix of the neutron and deuteron . According to the equalities (17), the “mixed” terms in Eq. (19) are equal to zero.

If the primary spin states of the neutron and deuteron are mutually independent ($\hat{\rho}_{nd} = \hat{\rho}_n \otimes \hat{\rho}_d$) , we obtain the following expression for the effective cross section of neutron elastic scattering on the deuteron :

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{nd} &= |\,a_{3/2}\,|^2\,\frac{2 + \mathbf{P}_n\,\mathbf{P}_d}{3} + |\,a_{1/2}\,|^2\,\frac{1 - \mathbf{P}_n\,\mathbf{P}_d}{3} = \\ &= \frac{2\,|\,a_{3/2}\,|^2 + |\,a_{1/2}\,|^2}{3} + \frac{|\,a_{3/2}\,|^2 - |\,a_{1/2}\,|^2}{3}\,\mathbf{P}_n\,\mathbf{P}_d \quad . \end{aligned} \tag{20}$$

Here

$$\mathbf{P}_n = tr\,(\,\hat{\boldsymbol{\sigma}}_n\,\hat{\rho}_n\,)\quad,\qquad \mathbf{P}_d = tr\,(\,\hat{\mathbf{s}}_d\,\hat{\rho}_d\,) \tag{21}$$

are the respective polarization vectors for the primary neutron and primary proton .

For the particles being unpolarized before scattering, we have :

$$\left(\frac{d\sigma}{d\Omega}\right)^{(0)}_{nd} = \frac{2\,|\,a_{3/2}\,|^2 + |\,a_{1/2}\,|^2}{3} \quad . \tag{22}$$

In this case the spin density matrix of the final particles after scattering takes the following form :

$$\hat{\rho}^{(\mathrm{fin})}_{nd} = \frac{1}{6}\;\hat{f}\;\hat{f}^{+}\left[\left(\frac{d\sigma}{d\Omega}\right)^{(0)}_{nd}\right]^{-1} = \tag{23}$$

$$= \frac{1}{6}\left(\,\hat{I}_n \otimes \hat{I}_d \;+\; \frac{|\,a_{3/2}\,|^2 - |\,a_{1/2}\,|^2}{2\,|\,a_{3/2}\,|^2 + |\,a_{1/2}\,|^2}\,(\,\hat{\boldsymbol{\sigma}}_n \otimes \hat{\mathbf{s}}_d\,)\right) \quad .$$

Let us introduce the correlation tensor of the final (*nd*) system with the components :

$$T_{ik} = tr\,(\hat{\sigma}_{n;\,i} \otimes \hat{s}_{d\,;\,k}\;\hat{\rho}^{(\mathrm{fin})}_{nd}\,) \quad . \tag{24}$$

Obviously, the sum of diagonal components of this correlation tensor amounts to:

$$T = \sum_{i=1}^{3} T_{ii} = tr\,(\,\hat{\boldsymbol{\sigma}}_n \otimes \hat{\mathbf{s}}_d\;\;\hat{\rho}^{(\mathrm{fin})}_{nd}\,) \quad . \tag{25}$$

Taking into account that the nonzero traces of product of matrices correspond to the projection squares :

$$tr\,(\hat{\sigma}_{n;\,i}\,)^2 = 2\quad,\qquad tr\,(\hat{s}_{d\,;\,k}\,)^2 = 2 \quad , \tag{26}$$

we obtain :

$$T_{ik} = \frac{|a_{3/2}|^2 - |a_{1/2}|^2}{2|a_{3/2}|^2 + |a_{1/2}|^2} \, \frac{2}{3} \, \delta_{ik} \quad . \qquad (27)$$

So, at $a_{1/2} = 0$ we have $T_{ik} = \frac{1}{3}\delta_{ik}$, $T = 1$;

at $a_{3/2} = 0$ we have $T_{ik} = -\frac{2}{3}\delta_{ik}$, $T = -2$.

These results are in full accordance with the values (13) .

4. Coherent length of neutron scattering on the deuteron

Taking into account the general formula (12), we obtain :

$$A_{coh}^{(nd)} = \frac{m_n + m_d}{m_d} \, \frac{4\, a_{3/2} + 2\, a_{1/2}}{2 \cdot 3} \quad , \qquad (28)$$

where m_n and m_d are the neutron and deuteron masses, respectively.

Since $m_d \approx 2\, m_n$,

$$A_{coh}^{(nd)} = \frac{3}{2}\left(\frac{2}{3}a_{3/2} + \frac{1}{3}a_{1/2}\right) = a_{3/2} + \frac{1}{2}a_{1/2} \quad .$$

According to the experimental data,

$a_{3/2} \approx 6.35$ Fm, $a_{1/2} \approx 0.65$ Fm, $A_{coh}^{(nd)} \approx 6.7$ Fm [1] .

5. Amplitudes of (*nd*)-scattering and (*nN*)-scattering

Since the deuteron represents the bound state of the neutron and proton, the question about the relation between the lengths of (*nd*)-scattering and (*nN*)-scattering on separate nucleons may arise. This problem could be solved simply, and it would be analogous to the problem of neutron scattering on the orthohydrogen molecule, if the approach based on the concept of Fermi pseudopotential [2] were valid in this case. Then, taking into account that, as it is well known, the triplet scattering length of free neutrons is equal to zero ($a_{t;\,nn} = 0$), we would obtain :

$$a_{3/2} = \frac{4}{3}\, a_{t;\,np} \quad ;$$

$$a_{1/2} = \frac{4}{3}\; \frac{a_{t;\,np} + 3\,(a_{s;\,np} + a_{s;\,nn})}{4} \;=\; \frac{1}{3}\, a_{t;\,np} + a_{s;\,np} + a_{s;\,nn} \qquad (\,?\,)$$

(here $a_{t;\,np}$ is the triplet length of (np)-scattering, $a_{s;\,np}$ and $a_{s;\,nn}$ are the singlet lengths of (np)- and (nn)-scattering, respectively) .

However, the method of Fermi pseudopotential is justified under the condition that the radius of a bound state is large as compared with the modulus of scattering amplitude. Only under this condition the binding energy, multiple scattering effects and the contribution of three-body forces may be neglected .

For the deuteron, this condition is surely not satisfied ; thus, at least, for the doublet length of (nd)- scattering the respective formula is wrong in principle .

But we would like to pay attention to the fact that in the case of quartet length of (nd)-scattering the situation is a bit different . Indeed, in the quartet state over the total spin of the (nd) system (J = 3/2) all the nucleon pairs are in the triplet state with the total spin 1. One may expect that the triplet length of neutron scattering on a bound neutron is very small, even if it does not equal zero . Then it is possible to disregard the multiple scattering of neutrons on the neutron and proton and, thus, the contribution into the quartet length of (nd)-scattering will be determined by the renormalized triplet length of neutron scattering on the bound proton :

$$a_{3/2} = \frac{4}{3}\, \tilde{a}_{t;\,np} \quad .$$

As follows from the experimental data, the effective triplet length of neutron scattering on the bound proton in the deuteron is by 10 % smaller than the triplet length of (np)-scattering on a free proton :

$$\tilde{a}_{t;\,np} \approx 0.89\, a_{t;\,np} \qquad (\; a_{t;\,np} = 5.38 \;\; \text{Fm} \;) \qquad .$$

6. Summary

1. The general structure of the *S*-wave scattering of hadrons on nuclei with arbitrary spin is analyzed.

2. The elastic scattering of slow neutrons on deuterons is considered. The operators of projection onto the quartet state (total spin J=3/2) and doublet state (total spin J=1/2) of the (*nd*) system and the scattering amplitudes, corresponding to these operators, are introduced. It is shown that in the low-energy limit the dependence of the effective cross section of (*nd*)-scattering is determined by only two scattering lengths (the quartet and doublet ones), which are summed up incoherently.

3. The quartet and doublet scattering lengths determine also the coherent length of neutron scattering on an unpolarized deuteron target.

4. The expression for the correlation tensor of the final (*nd*) system at the low-energy scattering of unpolarized neutron and deuteron is derived.

5. The question about the relation between the quartet length of (*nd*)-scattering and triplet length of neutron scattering on a proton is discussed.

References

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