PROPOSAL OF AN EXPERIMENT TO INVESTIGATE PROPERTIES OF THE NEUTRON WAVE-PACKET

V.K. Ignatovich

FLNP JINR

The concept of the coherence length of the neutron is explored. The generally accepted definition of a Gaussian wave packet based on the method of the beam preparation, and the singular de Broglie's wave packet are considered. Possible ways of measuring the coherence length are discussed.

Examination of wave packets is, apparently, the most important fundamental problem of physics today. It is clear that the wave function of the neutron is not a plane wave. It should be a vector of the Hilbert space i.e. a wave packet having certain properties. It has some size, called the coherent length, and the size can change with energy. In [1] it was reasonably noted that unstable particle with a lifetime of τ can have the size of the value of $L = \sqrt{\hbar\tau/m}$, which for the neutron is 0.75 cm. However, the finite lifetime can lead also to the definition of the packet size proportional to the neutron velocity $L = v\tau$. So the greater the speed, the larger is the wave packet, although from a physical point of view, it seems more reasonable to consider [2], that the faster the particle, the more neutron should look like a point particle.

In [3] wave packet was introduced to explain the UCN anomaly (abnormally high losses in traps). The packet size was estimated to be $L \approx 10^5\lambda_{\rm Be}$, where $\lambda_{\rm Be}$ is the minimal wavelength of the neutron which can be stored in beryllium traps. In this case L is the order of several millimeters, which does not contradict the assumption made in the [1]. Later [4,5] the assumption was made that the size of the wave packet depends on the speed and is proportional to the wavelength: $L \approx 10^5\lambda/2\pi$. Therefore, the thermal neutron wave packet size is up to about 10 microns. Perhaps it is correct, but how experimentally to measure the size of the wave packet — that is the question.

One way is to look for neutron transmission through films at incidence at a subcritical glancing angle [4,6,7]. Some indication of the transmission was received, but there is still no certainty. More precise experiments are needed.

Here another experiment is discussed. When, because of some coherent process, the neutron wave function of a polarized neutron is split into two diverging in the space oppositely polarized components, one can observe superposition of polarization and find the distance, at which superposition is terminated, i.e. two components diverge and the neutron becomes only in one of the components.

To predict theoretically transition of a coherent superposition into incoherent one, it is necessary to accept a model. Gaussian packets are not suitable because they spread out in the space. Gaussian packets are the result of the beam preparation. Their width Δk in momentum space characterizes the coherent length $l_c = 1/\Delta k$ in coordinate space. This length is well observed in experiments on interference measurements of distances [8–10]. We will accept a more appropriate model of a nonspreading singular de Broglie's wave packet

$$\Psi_{\rm dB}(r,\boldsymbol{k},t) = \sqrt{\frac{q}{2\pi}}\exp(i\boldsymbol{k}\boldsymbol{r} - i\omega t)\frac{\exp(-q|\boldsymbol{r}-\boldsymbol{k}t|)}{|\boldsymbol{r}-\boldsymbol{k}t|}, \tag{1}$$

wherein the spatial size is $l = 1/q$. To get rid of various constants, the variable t in (1) includes factor $\hbar/m$, so that t has dimension cm^{-2}, and the angular frequency is $\Omega = (k^2 - q^2)/2$. The packet (1) is normalized to unity by integration over the volume: $\int d^3r|\psi_{\ rmdB}|^2 = 1$, or as a full flux through any plane, integrated over time. Fourier decomposition of this packet has the form

$$\Psi_{\rm dB}(r, \boldsymbol{k}, t) = \sqrt{\frac{q}{2\pi}} \exp(i\boldsymbol{kr} - i\omega t) \frac{4\pi}{(2\pi)^3} \int d^3p \frac{\exp(i\boldsymbol{p}(\boldsymbol{r} - \boldsymbol{k}t))}{p^2 + q^2}, \tag{2}$$

and it follows that the packet satisfies the equation

$$\left(\frac{i\partial}{\partial t} + \frac{\Delta}{2}\right) \Psi_{\rm dB}(r, \boldsymbol{k}, t) = -q\sqrt{2\pi} \exp(-i(k^2 + q^2)t/2)\delta(\boldsymbol{r} - \boldsymbol{k}t). \tag{3}$$

In the next three sections some possible experiments with the de Broglie wave packet are discussed. Detailed calculations are presented in the appendices.

Measuring of the de Broglie wave packet size at coherent splitting of the neutron wave into 2 parts with equal but not collinear speeds. Consider a packet (1), which describes a particle with a fixed speed. Imagine a neutron flying at a speed of $\boldsymbol{k}$ along the axis x and polarized along the axis y. In some point $x = 0$, as shown in Fig.1, it is split into 2 components polarized along and opposite axis z and propagating at an angle θ to the x-axis. Spinor wave function of the neutron can be written as

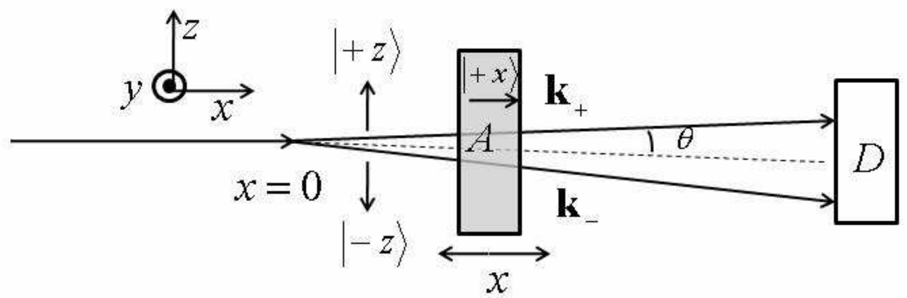

Fig. 1: *The neutron because of a coherent process is split into two oppositely polarized components symmetrically propagating at a small angle θ relatively to the original direction. Superposition of split component passes through an analyzer which transmits only neutrons polarized along the x-axis. The intensity recorded by the detector after analyzer shifting along the x-axis, should contain an oscillating component of the type shown in Fig.2.*

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(\Psi_{\rm dB}(r, \boldsymbol{k}_+, t)| + z\rangle + \Psi_{\rm dB}(r, \boldsymbol{k}_-, t)| - z\rangle\right), \tag{4}$$

where $\boldsymbol{k}_\pm = (k_x, 0, \pm k_z)$, $k_x = k\cos\theta$, $k_z = k\sin\theta \approx k\theta$. Let's put on the way of split neutron an analyzer transmitting only neutrons, polarized along the x-axis (it can be also along the axis y, which is not essential). Since $| \pm z\rangle = (|x\rangle \pm | - x\rangle)/\sqrt{2}$, then the wave function, transmitted by the analyzer is

$$\psi(\boldsymbol{r}, t) = \langle x||\Psi\rangle = \frac{1}{2} \left(\Psi_{\rm dB}(r, \boldsymbol{k}_+, t) + \Psi_{\rm dB}(r, \boldsymbol{k}_-, t)\right). \tag{5}$$

Therefore, the total flux through the plane y, z, located at the point x after the analyzer is

$$I(x) = \frac{1}{4}(I_+ + I_- + I_\pm), \tag{6}$$

where

$$I_{+,-} = k_x \int dydzdt \left|\Psi_{\rm dB}(r, \boldsymbol{k}_\pm, t)\right|^2, \tag{7}$$

$$I_\pm = 2k_x \int dydzdt \cos(2k_z z) \left|\Psi_{\rm dB}(r, \boldsymbol{k}_+, t)\Psi_{\rm dB}(r, \boldsymbol{k}_-, t)\right|. \tag{8}$$

The integrals (7), as is easy to show, are equal to one because of normalization. Of interest is the interference flux $I_\pm$. Detailed calculation is presented in the Appendix A. Result is represented by the function

$$f(X) = 2\eta \int_0^1 ds \cos(sX) \frac{\exp\left(-X\sqrt{1 - s^2 + \eta^2}\right)}{\sqrt{1 - s^2 + \eta^2}}, \tag{9}$$

on two dimensionless parameters $X = 2kx\theta^2$ and $\eta = q/k\theta$.

Note that the q can currently be estimated [3] by the value of $10^{-5}k$. If $\theta \approx 10^{-5}$ then $\eta \approx 1$, and the attenuation length is approximately equal to the period of oscillation, so the oscillations after analyzer virtually will not be observed. To observe them, it is desirable to have $\theta \approx 10^{-4}$. Then $\eta \approx 0.1$. With this setting of η the function (9) looks as shown in Fig. 2.

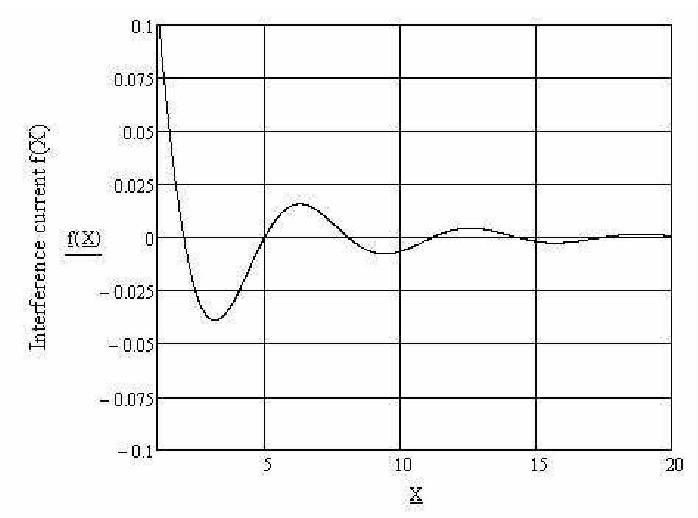

Fig. 2: *The result of a numerical calculation functions (9) for $\eta = 0.1$.*

The variable X corresponds to displacement in space

$$x = X\lambda/4\pi\theta^2, \tag{10}$$

where λ – neutron wavelength. For the experiment, the displacement x should be large enough. If the parameter is $X = 1$ corresponds to the actual displacement of 1 cm, then the $\theta \approx 10^{-4}$ neutron wavelength should be $\lambda \approx 10$ Å, which corresponds to an energy of 1 meV. Let's estimate how to get splitting of $\theta \approx 10^{-4}$.

Imagine transmission of a beam through a polarized magnetized prism, as shown in Fig. 3. The neutron beam polarized along the x-axis, falls from the left perpendicular to the vertical face of the prism magnetized along the axis z. Near the exit oblique face within the prism the wave vectors of the two components polarized along the z-axis are equal $\boldsymbol{k}_{in,1,2} = \boldsymbol{n}\sqrt{k^2 - u_{1,2}}\cos\phi + \boldsymbol{t}\sqrt{k^2 - u_{1,2}}\sin\phi$, where $\boldsymbol{n}$ and $\boldsymbol{t}$ are the unit vectors along the normal and along generatrix of the oblique edge, respectively, and $u_{1,2}$ are interaction potentials of the two spin components with prism material and its magnetic induction. After exiting the prism into space without magnetic field the wave vectors are $\boldsymbol{k}_{out1,2} = \boldsymbol{n}\sqrt{(k^2 - u_{1,2})\cos^2\phi + u_{1,2}} + \boldsymbol{t}\sqrt{k^2 - u_{1,2}}\sin\phi$. The square of the difference of two vectors,

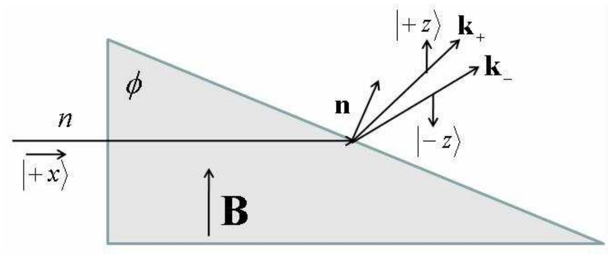

Fig. 3: *Getting slightly split beam of neutrons by transmission through a magnetized prism.*

divided by the square of the sum is equal to θ^2:

$$\theta^2 = \frac{\left(\sqrt{k^2 + u_1 \operatorname{tg}^2 \phi} - \sqrt{k^2 + u_2 \operatorname{tg}^2 \phi}\right)^2 + \left(\sqrt{k^2 - u_1} - \sqrt{k^2 - u_2}\right)^2 \operatorname{tg}^2 \phi}{\left(\sqrt{k^2 + u_1 \operatorname{tg}^2 \phi} + \sqrt{k^2 + u_2 \operatorname{tg}^2 \phi}\right)^2 + \left(\sqrt{k^2 - u_1} + \sqrt{k^2 - u_2}\right)^2 \operatorname{tg}^2 \phi}, \tag{11}$$

which for small u is reduced to

$$\theta = \frac{\Delta u}{4k^2} \operatorname{tg} \phi, \tag{12}$$

where $\Delta u = u_1 - u_2$. At $k^2 = 1$ meV the value $\theta = 10^{-4}$ is obtained, if $\Delta u = 10^{-7}$ eV, which corresponds to magnetization 2T, and $\operatorname{tg} \phi = 4$, 1.e. $\phi \approx 75^0$.

The result (9) is obtained for a fixed packet speed $k = \sqrt{k_x^2 + k_z^2} \approx k_x$. Let us now imagine that in fact we have a Gaussian distribution of speeds

$$w(k) = \frac{1}{\sqrt{\pi}\Delta} \exp\left(-\frac{(k-1)^2}{\Delta^2}\right), \tag{13}$$

where all the parameters are defined in terms of the average speed k_0. Integrating (9) over this distribution we get

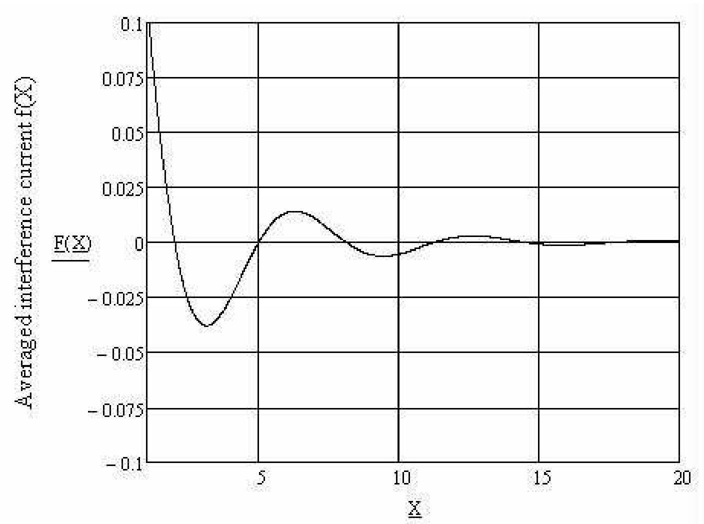

Fig. 4: *The result of the averaging of the function Fig.2, in accordance with (14) for* $\Delta = 0.1$.

$$F(X) = \int\limits_{-\infty}^{\infty} \frac{2k\eta dk}{\sqrt{\pi}\Delta} \exp\left(-\frac{(k-1)^2}{\Delta^2}\right) \int_0^1 ds \cos(sXk) \frac{\exp\left(-Xk\sqrt{1-s^2+\eta^2}\right)}{\sqrt{1-s^2+\eta^2}}. \tag{14}$$

The result for the function shown in Fig.2, is presented in Fig. 4 for $\Delta = 0.5$.

Scheme of the packet size measurement by splitting of the neutron into 2 components with different but parallel speeds

Suppose now that $\boldsymbol{k}_\pm = (k \pm \delta, 0, 0)$, where $\delta/k \ll 1$. Then (8) takes the form

$$I_\pm = 2k \int dydzdt \cos(2\delta x - 2k\delta t) \left|\Psi_d(r, \boldsymbol{k}_+, t)\Psi_d(r, \boldsymbol{k}_-, t)\right| . \tag{15}$$

The calculation of this integral, as detailed in Appendix B, again leads to the function (9), in which the dimensionless parameters are $X = kx\zeta^2 \quad \eta = q/k\zeta$, where $\zeta = \delta/k$ has the same role as the parameter θ in (9).

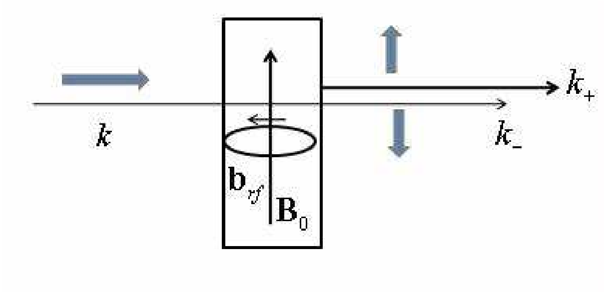

Fig. 5: *Getting a slightly longitudinally split beam of neutrons via RF spin-flipper [11].*

The experimental scheme is shown in Fig. 5. If neutrons of energy 10^{-4} eV pass through RF spin flipper with frequency 10 MHz, the velocities of the neutron components polarized up and down become different by an amount $\delta = k\hbar\omega/k^2 = 10^{-4}k$. The parameter $\eta = 0.1$, and the function (9) has the same form as shown in Fig. 2. Let's see what will be the oscillation period. Since it is determined by the same formula as the (10) only with the replacement of $\theta \to \zeta$, and the energy of 10^{-4} eV corresponds to $\Lambda \approx 30$ Å, the parameter $X = 1$ will correspond to $x \approx 3$ cm. Thus, the whole picture, as shown in Fig. 2 can be seen by moving the analyzer to 60 cm.

Scheme of the experiment for measuring the packet size by splitting of the neutron into 2 parts with different and non-parallel speeds.

Consider now the case when the neutron is split into 2 components propagating with different velocities at an angle to each other. Such a beam, for example, can be obtained by reflection of a neutron beam polarized along the external magnetic field from the magnetic mirror with magnetization noncollinear to the external field. At the reflection the reflected beam is split into 2 Components. One, reflected in the specular direction, retains its polarization and the velocity of the incident beam and the other has the opposite polarization and is reflected in the non-specular direction [12] under larger grazing angle.

We choose the x-axis in such a way that the components of the wave vectors have the form $\boldsymbol{k}_{1,2} = (k \pm \delta, 0, \pm k_z)$. Then the integral (8) becomes

$$I_\pm = 2k_x \int dydzdt \cos(2k_z z + 2x\delta - 2k\delta t) \left|\Psi_d(r, \boldsymbol{k}_1, t)\Psi_d(r, \boldsymbol{k}_2, t)\right| . \tag{16}$$

Obviously, the calculations result in the same function 9, but instead of the parameter θ there will be

$$\xi = \sqrt{\theta^2 + \zeta^2} = \frac{|\boldsymbol{k}_+ - \boldsymbol{k}_-|}{|\boldsymbol{k}_+ + \boldsymbol{k}_-|} \approx \frac{\Delta k}{k_0}. \tag{17}$$

Let's estimate the value of these parameters in the experiment [12]. Since the experiment was performed with thermal neutrons, and external field varied from 33 Oe to 4000 Oe, which corresponds to a magnetic energy from 10^{-10} and 10^{-8} eV, the parameter ξ changed in within $10^{-6} \div 10^{-8}$, and the parameter η in (9) was much greater than unity, then (9) can be approximated as

$$f(X) \approx \int_0^1 ds \cos(sX) \exp(-X\eta) \approx \exp(-2qx\xi) = \exp\left(-2 \cdot 10^{-5} kx\xi\right), \tag{18}$$

therefore coherence length, respectively, in this case

$$L_c = 10^5 \lambda / 4\pi\xi, \tag{19}$$

varies in the range $1 \div 100$ m, i.e. in this case practically we deal with a plane wave.

Conclusion This article shows the possibility of a direct measurement of the size of neutron wave packet, described by the singular de Broglie's function. No fundamental difficulties for corresponding experiments are expected.

Appendix The calculation of the integral (8).

We will show here in detail how to calculate the integral (8). We use the Fourier representation (2), write $2\cos(2k_z z)$ as the sum of two exponents and obtain the integral

$$I_\pm = \frac{q(4\pi)^2}{(2\pi)^7} \int \frac{k_x d^3p d^3p' dydzdt}{(p^2+q^2)(p'^2+q^2)} \exp(i\boldsymbol{p}(\boldsymbol{r}-\boldsymbol{k}_+t) + i\boldsymbol{p}'(\boldsymbol{r}-\boldsymbol{k}_-t) \pm 2ik_z z). \tag{20}$$

Integrating over the plane y, z and over time, and then over $\boldsymbol{p}'$, gives

$$J = \frac{q(4\pi)^2}{(2\pi)^4} \int \frac{d^3p \exp(-2i[p_z \pm k_z]xk_z/k_x)}{(p^2+q^2)\{p_y^2 + (p_z \pm 2k_z)^2 + (p_x + 2[p_z \pm k_z]k_z/k_x)^2 + q^2\}}. \tag{21}$$

We make the change of variables $p_z \pm k_z \to p_z$, and introduce $\theta = k_z/k_x \approx k_z/k$ then the integral is transformed into

$$J = \frac{q}{\pi^2} \int \frac{d^3p \exp(-2ip_z x\theta)}{(p_y^2 + (p_z \mp k_z)^2 + p_x^2 + q^2)\{p_y^2 + (p_z \pm k_z)^2 + (p_x + 2p_z\theta)^2 + q^2\}}. \tag{22}$$

Let us make more the change of variable $p_x + p_z\theta \to p_x$. Then the integral is transformed into

$$\int \frac{(q/\pi^2) d^3p \exp(-2ip_z x\theta)}{(p_y^2 + (p_z \mp k_z)^2 + (p_x - p_z\theta)^2 + q^2)\{p_y^2 + (p_z \pm k_z)^2 + (p_x + p_z\theta)^2 + q^2\}}. \tag{23}$$

We use the transformation

$$\frac{1}{AB} = \int_0^1 \frac{d\alpha}{(\alpha A + (1-\alpha)B)^2}. \tag{24}$$

Then the integral is transformed into

$$\int_0^1 d\alpha \int \frac{(q/\pi^2) d^3p \exp(-2ip_z x\theta)}{(p_y^2 + p_z^2(1+\theta^2) + p_x^2 + 2(p_x p_z \theta \pm p_z k_z)(1-2\alpha) + k_z^2 + q^2)^2}. \tag{25}$$

We make the change of variables $s = 2\alpha - 1 \;\; p_x - sp_z\theta \to p_x$. Then (25) is transformed into

$$J = \int_{-1}^{1} ds \int \frac{(q/2\pi^2) d^3p \exp(-2ip_z x\theta)}{(p_y^2 + p_x^2 + p_z^2(1 + (1 - s^2)\theta^2) \mp 2p_z k_z s + k_z^2 + q^2)^2}. \quad (26)$$

Integrating over dp_x, dp_y gives

$$J = \int_{-1}^{1} ds \int \frac{(q/2\pi) dp_z \exp(-2ip_z x\theta)}{p_z^2(1 + (1 - s^2)\theta^2) \mp 2p_z k_z s + k_z^2 + q^2}. \quad (27)$$

For small θ integral can be simplified

$$J \approx \int_{-1}^{1} ds \int \frac{(q/2\pi) dp_z \exp(-2ip_z x\theta)}{p_z^2 \mp 2p_z k_z s + k_z^2 + q^2}. \quad (28)$$

We make the change of variables $p_z \mp sk_z \to p_z$, then we get

$$J = \int_{-1}^{1} ds \int \frac{(q/2\pi) dp_z \exp(\mp 2isk_z x\theta) \exp(-2ip_z x\theta)}{(p_z^2 + (1 - s^2)k_z^2 + q^2)}. \quad (29)$$

You can now integrate over p_z. The result is

$$I_\pm = 2q \int_{0}^{1} ds \cos(2skx\theta^2) \frac{\exp\left(-2x\theta\sqrt{q^2 + k_z^2(1 - s^2)}\right)}{\sqrt{q^2 + k_z^2(1 - s^2)}}. \quad (30)$$

Take out k_z outside square root. As a result, we obtain the function (8).

The calculation of the integral (15) Write $2\cos(\vartheta)$ as a sum of two exponents, then we get the integral

$$J = \frac{q(4\pi)^2}{(2\pi)^7} \int \frac{kd^3pd^3p'dydzdt}{(p^2 + q^2)(p'^2 + q^2)} e^{i\boldsymbol{p}(\boldsymbol{r} - \boldsymbol{k}_+ t) + i\boldsymbol{p}'(\boldsymbol{r} - \boldsymbol{k}_- t) \pm 2i\delta x \mp 2ik\delta t}. \quad (31)$$

Integration over time and coordinates and leads to three δ-functions which make it easy to integrate over d^3p', resulting in the expression

$$J \approx \frac{q}{\pi^2} \int \frac{d^3p \exp(\mp 2ix\delta^2/k) \exp(-2ix\delta p_x/k)}{(p^2 + q^2)(p_y^2 + p_z^2 + [p_x \pm 2\delta]^2 + q^2)}. \quad (32)$$

Change of variables $p_x \pm \delta \to p_x$ leads to

$$J \approx \frac{q}{\pi^2} \int \frac{d^3p \exp(-2ix\delta p_x/k)}{(p_y^2 + p_z^2 + [p_x \mp \delta]^2 + q^2)(p_y^2 + p_z^2 + [p_x \pm \delta]^2 + q^2)}. \quad (33)$$

We make transformation (24) and change of variables $2\alpha - 1 = s$. As a result we obtain

$$J = \int_{-1}^{1} \frac{dsq}{2\pi^2} \int \frac{d^3p \exp(-2ix\delta p_x/k)}{(p_y^2 + p_z^2 + p_x^2 + q^2 + \delta^2 \mp 2sp_x\delta)^2}. \quad (34)$$

Change of variable $p_x \mp s\delta \to p_x$ and integration over $dp_y dp_z$ gives

$$J = \int_{-1}^{1} \frac{dsq}{2\pi} \int \frac{dp_x \exp(\pm 2isx\delta^2/k) \exp(-2ix\delta p_x/k)}{p_x^2 + q^2 + \delta^2(1 - s^2)}. \tag{35}$$

After integration over dp_x we finally obtain the function (9), in which $\eta = q/k\zeta$, $\zeta = \delta/k$ and $X = 2xk\zeta^2$.

References

[1] A.Steyerl, S.S.Malik, "Possible implication of exponential decay", Annals of Physics **217**222-278 (1992)

[2] Ignatovich V.K. "On a possibility of continious transition from wave optics to geometrical one." JINR communication E4-8039 (1974).

[3] Ignatovich V.K., Utsuro M. "Tentative solution of UCN problem" Phys. Lett. A. **225** 195–202 (1997)

[4] Utsuro M., Ignatovich V.K. "Experimental test of the de Broglie wavepacket description of the neutron" Phys. Lett. A. **246** 7–15 (1998)

[5] M.Utsuro, "Analytical solution for the de Broglie wave packet description of the neutron in subcritical transmission through a mirror."Physics Letters A **292** 222-232 (2002)

[6] V. K. Ignatovich and Masahiko Utsuro, "Quantum mechanics of the de Broglie wave packet and a review of inelastic losses of UCN in bottle," Nucl. Instr. & Meth. A **440** 709-716 (2000)

[7] M.Utsuro, "Neutron spin interference visibility in tunneling transmission through magnetic resonators," Physica B **358** 232-246 (2005)

[8] F.V. Ignatovich, D.S. Gibson and M.A. Marcus, "Apparatus and method for evaluation of optical elements," U.S. Patent 9, 019, 485 B2. April 28, 2015.

[9] M. A. Marcus, J-R Lee, and H.W. Harris, "Method for Measuring Material Thickness Profiles," U.S. Patent 6,038,027, 2000.

[10] M. L. Dufour, G. Lamouche, S. Vergnole, B. Gauthier, C. Padioleau, M. Hewko, S. Levesque, and V. Bartulovic, "Surface inspection of hard to reach industrial parts using low-coherence interferometry", Proc. SPIE Int. Soc. Opt. Eng., 6343, 63431Z, 2006

[11] M.Utsuro, V.K.Ignatovich "Handbook of neutron optics". Wiley-VCH, Berlin. 2010.

[12] Y.V. Nikitienko, V.K. Ignatovich, S.V. Kozhevnikov, A.V. Petrenko, Two-mirror neutron spin interferometer, to be published.