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Neutron Slowing Down as a Random Walk Problem

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In this report author asserts that the names of Pierre-Simon Laplace, Gian-Carlo Wick, Enrico Fermi, Karl Pearson and Paul Langevin happened to be directly related to the historical problems of the random walk in the statistical theory and the slowing down of neutrons to thermal energy in the physics. The works of these authors have been instrumental in solving the mentioned problems by developing the exact mathematical expressions for the probability density of the sum of independent random variables. Several approaches to obtain these expressions will be shown. The most problematic were difficulties of getting the result in the form of an analytical formula for energy distribution of neutrons, which are slowed down by a fixed numbers of impacts with protons. The author's simplest way to deduce such formula is shown also, for the pedagogical reasons, for students interested in neutron physics.

1. Introduction

Random walk is a traditional subject of the mathematical statistics initially introduced by Karl Pearson in 1905 [1] who defined it as sequences of '*n*' steps taken in random directions of space, often with a random step length too. The problem to solve was to find the probability of "landing" at a given spot of the space after a giving number of steps. Both, the directions and the length of a particular step have to be chosen from a given probability distributions and are independent of these quantities for other steps. The mathematical formalism of random walks is often formally identical for numerous problems in physics, in particular for the probabilistic treatment of the neutron diffusion during the neutron movement in the state of the thermal equilibrium with a nuclei media after the fast neutrons are slowing down below the energy of about 1 eV. Here the neutron-nuclei collisions make the diffusion motion appear to be a random walk. The mean velocity of neutrons in this case is constant, but the time and the space are the random variables. In our report, we will discuss another problem – the problem of the fast neutrons (several MeV of the initial energy) slowing down by the elastic collisions with protons in an infinite hydrogenous media. At each collision neutrons lose a part of their remaining kinetic energy or even transfer all energy to nuclei. For this case, the random variables in question are the neutron energies during the neutron free *flights* between subsequent collisions and the number of collisions, '*n*', leading to a given neutron energy. The random walk formalism assumes the probabilistic description of the slowing down phenomenon. The mean values and the dispersions around the mean for the quantities under study are obtained after deducing the proper probability distribution functions.

2. Neutron slowing down problem ("Fermi and Wick")

In the April of 1935 the Fermi's group published [2] their most interesting result on the effect of an enormously increasing radioactivity of their samples, when the fast neutron source was surrounded by paraffin or water. The following are quotations from this paper: "We will give an interpretation of this phenomenon by assuming that neutrons are slowed down by impacts against hydrogen nuclei", and more: "It is *easily shown* that an impact of a neutron against a proton reduces, on the average, the neutron energy by a factor of $1/e$". Continuing this subject, Gian-Carlo Wick published, about half a year later, his single author paper [3], writing in the beginning: "According to reports from several sides, the above passage in a paper by Professor Fermi is considered somewhat *obscure*. Since a more detailed explanation might be of interest also to others, it was thought advisable to make it generally known."

Wick begins that, from one side, it is follows from the neutron energy and momentum conservation in collisions with the isotropic scattering in CM-system, that the probability for the scattering from any energy E to energy E' is the uniform distributed function $1/E$ (independent of energy E'), see also e.g. [4]. Therefore the mean neutron energy after one collision, $<E'>$, is equal to one half of the neutron energy E before the collision: $<E'>=(E/2)$. Then after n sequential collisions, started with the fast neutron initial energy E_o (for example, E_0 = 2 MeV), the 'final' neutron energy E_n is expected to be $<E_n> =E_0/(2)^n$, on the average. This is not in accord with the Fermi's factor of $1/e$.

From the other side, Wick introduces the logarithmic variable $ln(E/E')$, known as the neutron energy decrement (the *lethargy*, presently) after one collision. Being a member of the Fermi's close circle, he had knew, that the Fermi's argument was based on the consideration of this variable. Because the logarithmic energy decrement is the sum of the partial decrements due to each collision, it is subjected to the Bernoulli's theorem of large numbers, so it will be possible to decide whether the mean energy is also the most probable one, or not. Defining the logarithmic random variable $x=ln(E_0/E_n)$ after n collisions, Wick writes in conclusion: "All further possible doubts may perhaps be answered by giving the entire probability distribution function. One finds:

$$f_n(x) = [x^{n-1}/(n-1)!]\, e^{-x}.\text{"} \tag{1}$$

It needs to keep in mind, that this Wick-Fermi probability density is the two-parameter function, and one hast to fix some value of the parameter n in order to have the energy distribution (in the ln scale) for this particular value of n. With this probability law, the expression $f(x)dx$ means the probability that the logarithmic variable x lies between x and $x+dx$. It is also follows that $<x> = n$, and setting n=1 it confirms the Fermi's $1/e$ statement as the statement for the *logarithmic* energy. Of course, the difference between factors $1/2^n$ and $1/e^n$ is of a little importance in many applications. Evidently however, that it was a matter of principle for Fermi to use the logarithmic variable to describe the randomness of a neutron's energy as it slows down, and it was namely Wick who introduced *into print* for the first time the logarithmic energy decrement. The Wick's Letter to the Editor of the Physical Review, however, does not contain the derivation of formula (1), which we will discuss as the Wick-

Breit formula, because the first published derivation of it appeared in the work by Edward Condon and Gregory Breit [5], as is reviewed in Section 3.

Concluding the present Section, let's remark that in the course of time other physicists worked on this problem using the random work theory as well as the Boltzmann transport equation. Sometimes, the probabilistic approach was contrasted to the Boltzmann's transport theory in the interpretation of the neutron slowing down results, though finally all agreed that the physical results are the same. The latest derivation, to our knowledge, was performed in the 1970's by C.S. Barnett [6], who obtained the analytical expressions for the mean, variance and even higher moments of the probability distribution function by using the modern methods of the Random Walk problem. For the average fission neutron initial energy of E_0 = 2 MeV and the 'final energy' of the slowing down phase E_n = 1 eV, Barnett finds $\langle n \rangle = 15.5\pm3.8$, to be compared with $\langle n \rangle = 20.9$ using the arithmetic average, and the value of the most probable number of collisions to the energy 1 eV, is $n_{\mathrm{peak}} = 14.5$.

3. Distributions of sums of neutron random variables ("Laplace, Condon-Breit")

There are similarities between papers of Wick [3] and of Condon, Breit [5]. The latter authors stated at the beginning (the quote): "The work grew out from a desire to understand statement due Fermi starting with: "It is *easily shown*…," − the full statement is quoted already in the precedent Section. Initially, Breit and Condon have used the arithmetic random variable: the fractional ratio of energies $\boldsymbol{x_i} = E_i/E_0$ and suggested to use the Laplace probability distribution function for the sum of such random variables, obtained at the end of the18th century. However, they soon switched to the logarithmic random variable $u=u_1 + u_2 + \ldots$, by using $\boldsymbol{x} = exp(-u)$, that is $-u = ln(x)$. One has to have in mind, that probability density functions for the individual logarithmic variables u_i are not *uniform* but are *exponential* functions $f_U(u_i) = \exp(-u_i)$.

Laplace [7] makes use of a uniform function $f_n(x)$ which is, in modern notation, written usually in the form of the following sum:

$$f_n(x) = [1/a^n(n-1)!]\{x^{n-1} - C_1^n(x-a)^{n-1} + C_2^n(x-2a)^{n-1} - \ldots.\}.$$

Here x is the random variable, i is the summation index, C_i^n are the Bernoulli, that is, the binomial coefficients. The expression $f_n(x)dx$ gives the probability that *the sum* of n elements each taken at random from a given range 0 to a (a >0) of *uniform* distribution, will fall into the interval from x to $x+dx$. Laplace applied it to the study of different inclinations of the planets to ecliptic in astronomy. This is exact mathematical expression for the probability density of the sum of random variables, having the same uniform distribution each. However, it is presented as a series of the numerous terms, instead of an analytical formula, which is difficult to evaluate without computers. Nevertheless, during the early development of the theory of probability, a great amount of work was devoted to the study of the probability distribution of such sums. Looking back one may say that these studies became a starting point of the works by which the modern Probability theory was created. With the logarithmic variable, being smart enough, using recursion technique, Breit and Condon worked out solutions of the problem in analytic form for slowing down neutrons by protons:

$$f_n(u)du = e^{-u}u^{n-1}du/(n-1)! \quad \text{and} \tag{2}$$

$$f_n(x)dx=(ln(1/x))^{n-1}dx/(n-1)!$$

One has to keep in mind that quantities x in the Fermi-Wick and the Breit-Condon formulas are reciprocal numbers, by definitions. Finally, Condon and Breit presented Figures for several values of n showing how rapidly the neutron energy is reduced with number of impacts n increasing.

4. Langevin's approach

When the German and French governments were seeking for a name for the neutron center at Grenoble, they decided to honor scientists for their contribution not only for science but to the society in general. The Grenoble international neutron center was named the Institute Laue Langevin. Not being a nuclear nor neutron physicist, Langevin started to work on the moderation of the fast neutrons in media composed by nuclei heavier then hydrogen, following the Joliot-Curie's suggestion at the end of 1940, when the German occupants imprisoned Langevin in Paris. He successfully concluded this study by several publications through 1941−1942 from the town of Troyes, where the Gestapo put him under the surveillance. These works became to be Langevin's last ones, performed shortly before his death. As before for neutron moderation by protons by other authors, Langevin calculated the probability for a fast neutron to reduce its energy to E, $E+dE$ by collisions with the any media nuclei after one, two, three, or any number of successive impacts.

Starting from an neutron energy ratio $C=E/E_0$ as an independent random variable, Langevin changed it to the logarithmic variable: $C=exp(-x)$, or $x=ln(E_0/E)$ [8]. Using the theorem of the addition of probabilities, he obtained the total probability P after n collisions as the sum $P = \sum_i P_n$.

Here P_3, for example, is the probability for $E=E_3$ to be in the interval ΔE when E_1 and E_2 impact does not happened. Langevin developed the original approach to represent probabilities and calculate them, which he named the "graphical, geometrical" one. He gave the Figures, for example, for the evaluation of the P_2 – the value of the probability to obtain energy E_2 after the 2nd impact of neutron with any nucleus. With his method Langevin independently deduced the already know for us the Laplace equation

$$f_n(x) = 1/[1/\{(n-1)!(1-\alpha^2\ )n]\}\{x^{n-1}- C_1^n(x-a)^{n-1} + C_2^n(x-2a)^{n-1} - \ (-)^i\ \ C_i^n(x-ia)^{n-1}\}.$$

Here the new parameter is $\alpha = (M - m)/(M + m)$, where M is the mass of the colliding nucleus, m − the neutron mass and the parameter a is $a = 2ln(1/\alpha)$. Langevin presented the analysis of this formula for several cases in his talks to the French Academy of Science at the several Sessions: Compt. Rendes Acad. Sci., Paris, v. 214, pages 517, 867, 889 (1942).

5. Summary

We have reviewed historical papers on the problem of the distribution of the sums of the independent random variable, solved for the first time by Laplace (18th century) and on the moderated neutron's energy spectra studied in 1930's − 1940's. The problem of the neutron impacts by nuclei was treated probabilistically. The result was given for the energy distribution of neutrons slowing down as they scatter multiple times by a hydrogenous or by more heavy media. This was done by G.C. Wick and by E.U. Condon, G. Breit. They introduced into print for the first time the neutron logarithmic energy decrement, known today as the *lethargy* variable, and gave solution in the analytical form for the case of impacts by protons. The great French scientist Paul Langevin solved the problem of the neutron moderation in a heavier then hydrogenous medium by an original, his own, geometrical method.

The author's pedagogical derivation of the analytical formula for the probability density of the sum of *n* logarithmic random variables follows below in Appendix.

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Appendix

By the definition, the probability density function of a *sum* of two independent random variables is the *Convolution* of their individual probability density functions. For a sum of *n* random variables a direct way is to perform the n-Fold Convolution, which is a fairly long procedure. Therefore, the author prefers to use the method of the mathematical *Induction*. It has three steps:

- ✓ Step 1: the *Induction Basis*. It was already stated in Section 3 that probability density functions for the individual logarithmic variables u_i are *exponential* functions $f_U(u_i)= exp(-u_i)$, and we will use this probability density for the n=1 variable and for other individual random variables by writing exponential $f(z) = exp(-z)$. In order to 'guess' the final *analytical* expression, we have also performed, as a preliminary step, convolutions for sums $Z2= f_{Z1}(z)+f_{Z1}(z)$, $Z3=f_{Z2}(z)+f_{Z1}(z)$, $Z4=f_{Z3}(z)+f_{Z1}(z)$. The obtained results are: $Z2 = z\, exp(-z)$, $Z3 = (1/1\cdot 2)\, z^2 exp(-z)$, $Z4 = (1/1\cdot 2\cdot 3)\, z^3\, exp(-z)$. The evidently visible patterns in these results, allows, preliminary, to write analytical expression for the sum of random logarithmic variables for the next Induction step.
- ✓ Step 2: the *Induction Hypothesis*. With the patterns found above, we assume, that the sum of the $(n-1)$ variables has the probability density function $f_{Z(n-1)}(z) = 1/(n-2!)\, z^{n-2}\, exp(-z)$. The goal of the next step will be to poof that $f_{Zn}(z)$, which, by definition, is the sum $f_{Z(n-1)}(z) + f_{Z1}(z)$ is described by the same analytical function but with the parameter value n.
- ✓ Step 3: the *Induction step*. To proof, we need to calculate the convolution of two functions:

$$f_{Zn}(z) = \int_0^\infty f_{z1}(z-s) f_{z(n-1)}(s) ds \;=\; \int_0^z exp\big(-(z-s)\big)\, z^{n-2}/\big((n-2)!\big) exp(-z) ds \;=$$

$$=exp(-z)/((n-2)!)\int_0^z s^{n-2}\, ds = exp(-z)/((n-2)!(n-1))z^{n-1} = exp(-z)/((n-1)!)z^{n-1}. \qquad (3)$$

Conclusion: the induction method gave us the formula (3), which is the same as expression (1) of Wick in Section 2, or the expression (2) of Condon and Breit in Section 3 as well.

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